With the kind regards
of
The Authors

A

TREATISE

ON

ALGEBRA

BY

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OF

CORNELL UNIVERSITY.

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PREFACE.

In writing this treatise on algebra, the authors have had two rules for their guidance.

As to matter: "Assume no previous knowledge of algebra, but lay down the primary definitions and axioms, and, building on these, develop the elementary principles in logical order; add such simple illustrations as shall make familiar these principles and their uses."

As to form: "Make clear and precise definition of every word and symbol used in a technical sense; make formal statement of every general principle, and, if not an axiom, prove it rigorously; make formal statement of every general problem, and give a rule for its solution, with reasons, examples, and checks; add such notes as shall indicate motives, point out best arrangements, make clear special cases, and suggest extensions and new uses."

In working out the plan here outlined, wide departures have been made from the standard text-books. Many new things have been introduced, not, indeed, because they were new, but necessarily, either as definitions in giving larger meanings to old words, or as axioms and theorems in stating and proving the elementary principles, or as problems and notes in showing new uses of principles already proved: *e.g.*, many fundamental principles were found to be omitted by elementary writers because too difficult for a beginner, and by subsequent writers as already known. A typical case is that of logarithms: that "the product of two powers of any same base is a power of that base whose exponent is the sum of the exponents of the factors" is generally proved for commensurable powers, but assumed, without proof, for incommensurable powers; and the whole theory of logarithms, so important, and their use, so common, are thus left to rest on faith. In a few cases new words and new symbols

have been introduced: notably the signs $^+$, $^-$, $^\pm$, and the copulas $\geqslant$, $\leqslant$, $\equiv$. It is believed that the need will justify the innovation.

Moreover, the tendency of modern work is to change the traditional boundaries of algebra so as to utilize graphic representation, the elements of infinitesimal analysis, and the calculus of operations, and only thus can the subject be presented most naturally and philosophically. A good example is that of the so-called "imaginaries," which, rightly presented, are as real as any other numbers.

The authors set out to write a text-book for the use of their own classes in the University, *i.e.*, for young men who had already studied the elements of algebra and geometry, and who had had some scholarly training; and, though an elementary book, at no time have they thought of it as a book for beginners. The wants of their classes have ever been before them; but the work has grown upon their hands until it embraces many topics that, from their nature or their treatment, are quite beyond the range of ordinary college instruction. As a text-book, therefore, for use in ordinary classes it must be abridged; yet its wide range makes it all the more valuable to teachers, as a book of reference, and to those bright scholars who wish either to place their knowledge of algebra on a sure foundation, or to make that knowledge a stepping-stone to the higher analysis.

Many thanks are due to Mr. James McMahon and Mr. A. S. Hathaway, instructors in mathematics in the University, for their very valuable assistance in the preparation of the text, and to Mr. Albert Jonas and Mr. E. C. Murphy, for useful suggestions, and for help in verification of the text and in proof-reading.

The careful reader will find many errors in this first edition; he will confer a favor if he will kindly communicate them to either of the authors. Any suggestion looking to the improvement of the book in either matter or form will be welcome.

Another edition will contain chapters on: theory of equations, integer analysis, symbolic methods, determinants and groups, probabilities, and insurance; with a full alphabetical index.

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ALGEBRA.

I. PRIMARY DEFINITIONS AND SIGNS.

ALGEBRA is that branch of Mathematics which treats of the relations of numbers. It is distinguished from Arithmetic, as having wider generalizations, as using signs and letters more freely, and as recognizing negatives and imaginaries. The applications of many words common in Arithmetic are greatly extended in Algebra, and their definitions are correspondingly enlarged.

The symbols explained below constitute a symbolic language, a species of short-hand writing, wherein numbers and their relations are more conveniently expressed than in the ordinary language of words. In this language the signs stand for words and phrases, and generally have the same grammatical relations as the words and phrases themselves. The words may be restored at any time. The reader should constantly practice translating from one form to the other till both are familiar.

This symbolic language is one of the characteristic features of Algebra; and among its many advantages are these: clearness, brevity, and generality of statement; the ability to mass directly under the eye, and thus to bring before the mind as a whole, all the steps in a long and intricate investigation; and the facility of tracing a number through all the changes it may undergo. Some other sciences, for example Chemistry and Logic, have a symbolic language of their own.

§ 1. NUMBER.

In measuring anything, some *unit* of the same kind is first assumed, and the relation the thing measured bears to this unit, *i.e.*, the operation that if performed upon the unit will produce the given thing, is expressed by a *number*. The unit, being acted upon, is the *operand*, the number is the *operator*, and the thing produced is the *result*.

Such numbers are also called *abstract numbers*, because all their properties and relations are independent of the particular units used; and the units and the measured things are *concrete numbers*. Abstract and concrete numbers are also called *quantities*.

Abstract numbers likewise arise from the combination of other abstract numbers: and in this way, their relations form the chief subject-matter of Algebra.

Two abstract numbers are *equal* if, operating upon the same unit in the same way, they produce the same result.

An abstract number is an *integer* if the thing measured be made up of entire units; a *simple fraction*, if the thing be one or more of the equal parts that the unit may be divided into. Integers, simple fractions, and such other numbers as can be reduced to integers or simple fractions, are *commensurable numbers:* those which cannot be so reduced are either *incommensurables* or *imaginaries*.

§ 2. REPRESENTATION OF NUMBERS.

Numbers are represented by Arabic numerals, or by letters. Among the more common forms are these:

$0, 1, 2, 3, \ldots, 10,$ read: *naught*, *one*, *two*, *three*, ..., *ten;*

$\alpha, \beta, \gamma, \delta, \epsilon, \theta, \pi, \phi, \Delta, \Sigma,$ read: *alpha*, *beta*, *gamma*, *delta*, *epsilon*, *theta*, *pi*, *phi*, *large delta*, *large sigma;*

$\frac{4}{7}, \frac{x}{y}, \frac{\pi}{2},$ read: *four sevenths*, *x over y*, *half pi*.

$a', b'', c^{iv}, d^{(m)},$ read: *a prime*, *b second*, *c fourth*, *d m^th^;*

$p_0, p_1, p_x,$ read: *p sub zero*, *p sub one*, *p sub x*.

The accents, numerals, and letters, attached to other numerals and letters, are *indices*. An index attached below its letter is a *suffix* or *subscript*, and is read *sub*. The index of a power [§ 10] is an *exponent*. The letter or numeral to which the index is attached is the *stem*. Sometimes the indices are written without the stem; this form of writing is called the *umbral notation*.

E.g., instead of $a_{1,2}$; $a_{3,4}$; ... $a_{i,k}$, write 1, 2; 3, 4; ... i, k.

The accent and subscript notation has two chief advantages:

It gives a very great number of distinct symbols, — an infinite number.

It permits numbers of the same kind to be represented by the same letters.

E.g., p', p'', p''', or p_1, p_2, p_3, may stand for the principals of three promissory notes;

then t', t'', t''', or t_1, t_2, t_3, will naturally stand for the three times for which these three notes are respectively given,

and r', r'', r''', or r_1, r_2, r_3, for the three rates.

The *value* of a letter or other symbol is the number for which it stands. Ordinarily the same letter stands for but one number during any one investigation, but for different numbers in different investigations; and different letters, or the same letter with different indices, for different numbers in the same investigation.

§ 3. POSITIVE AND NEGATIVE NUMBERS.

WHEN the measuring unit is taken in the same sense as the quantity measured, the number is *positive;* when in the opposite sense, the number is *negative*. In which sense the unit shall be taken, is a matter of custom or convenience.

Manifestly, if two quantities, opposite in sense, are measured by the same unit, one number is positive and the other negative:

E.g., if distances to the north or east from a given point are positive, distances to the south or west are negative: *i.e.*, if the measuring unit is a northerly or easterly unit, then southerly or westerly distances are expressed by negative numbers, and vice versa.

So, if the revolutions of a wheel forward are positive, revolutions backward are negative;
if assets are positive, liabilities are negative;
if dates A.D. are positive, dates B.C. are negative;
if the readings of a thermometer above zero are positive, the readings below zero are negative.

The primary notion of a negative number is that of one which, when taken with a positive number of the same kind, goes to diminish it, to cancel it altogether, or to reverse it.

E.g., liabilities neutralize (negative) so much of assets, thereby diminishing the net assets or leaving a net liability.

If two numbers of the same kind, when taken together, exactly cancel each other, they are *opposites*, one of the other.

Manifestly, of two opposites, one is positive and the other is negative.

So, if numbers are used as indices of two algebraic operations which when performed successively tend to neutralize each other, a positive number is commonly used for one index and a negative number for the other; and sometimes, as with exponents of powers [§ 10], custom has permanently determined which index shall be positive and which negative.

When denoted by Arabic numerals, positive numbers are written with the sign $+$ or with no sign, and negative numbers with the sign $-$, and it is evident at sight whether the number is positive or negative.

E.g., if the measuring unit be \$1 of assets, then $+100$, or simply 100 without the sign, expresses the net value of an estate whose assets exceed its liabilities by \$100; and -100, that of an estate whose liabilities exceed its assets by \$100.

But, if a number be denoted by a letter, it is not evident upon its face, and often it is not necessary to know, whether that letter denotes a positive or a negative number.

E.g., in the above example, N may stand either for $+100$ or for -100, at the pleasure of the writer. If, however, N stands for $+100$, then $-\text{N}$ stands for -100; and if N stands for -100, then $-\text{N}$ stands for $+100$. In either case $+\text{N}$ and $-\text{N}$ are opposites.

In this use of the signs $+$ and $-$, they are called *signs of quality*, since they indicate the quality, in an important particular, of the quantities measured, and of the numbers before which they stand.

These signs are also used to indicate the operations of addition and subtraction [§§ 6, 7], and are then called *signs of operation;* but, as the reader will see when he comes to the study of these operations, the two uses are always in accord, and the signs may often be understood in either way at pleasure. Sometimes signs performing both offices occur before the same number [§§ 6, 7].

The sign $^+$ before a number denotes either the number itself, or its opposite, whichever of them is positive; the sign $^-$ denotes whichever of them is negative; *i.e.*, a number preceded by $^+$ is essentially positive, and a number preceded by $^-$ is essentially negative.

E.g., if N stands either for 100 or for -100, $^+$N, read N *taken positive*, stands for $+100$; and $^-$N, read N *taken negative*, stands for -100.

So, $^+100$ may always be written for $+100$, and $^-100$ for -100; but not $^+$N for N or $+$N, nor $^-$N for $-$N, unless the value of N be positive.

Manifestly, $^+$N and $^-$N are opposites; and so are $^+100$ and $^-100$.

NOTE. — The reader should observe that some things admit of negatives and some do not.

E.g., time may be counted backwards as well as forwards from any given date; so may distance from any given point; so may heat and cold from an arbitrary zero; so may money of account, as above; but with real dollars, say five of them, he will find, when he tries to count past none, — five, four, three, two, one, none, — that he is attempting to do what is impossible.

So, when he comes to the study of the so-called imaginaries, he will find that for some things they have a real existence, but for other things they have not.

So, for some things, fractions have no existence.

E.g., $\frac{1}{2}$ of a man, or $\frac{2}{3}$ of an atom, or $1\frac{1}{2}$ events or facts, would be unmeaning.

§ 4. SPECIAL SIGNS.

THE *sign of continuation* is $\ldots$, read *and so on.*

E.g., $1,\ -2,\ +3,\ \ldots,\ +9$
means $1,\ -2,\ +3,\ -4,\ +5,\ -6,\ +7,\ -8,\ +9.$

The *signs of inference* are $\because$, read *since* or *because*, and $\therefore$, read *therefore*.

E.g., $\because$ 80 cts. < \$1, $\therefore$ 400 cts. < \$5;
or 400 cts. < \$5, $\because$ 80 cts. < \$1.

The *signs of grouping* are ( ), [ ], { }, $\overline{\quad}$, | . They show that all within the *brackets*, under the horizontal *bar*, or before the vertical *bar*, is taken together as one number, and subject to the same operation; viz., that which is indicated by the sign preceding or following it, or by the index attached to it.

When two or more numbers joined by the signs + and − [§§ 6, 7] are grouped together by a bar or brackets, they form an *aggregate*.

E.g., $(1+2+3)\times\overline{5-2}$ is the product of two aggregates [§ 8].

When two statements are identical, except only for a few characteristic words or signs, then, as a matter of convenience, the two statements may be written together as a *double statement*, by placing the pairs of corresponding words or signs one above the other.

E.g., $\because$ the battle of Salamis was fought 480 B.C. and that of Waterloo 1815 A.D.,

$\therefore$ $\begin{matrix}\text{Salamis}\\ \text{Waterloo}\end{matrix}$ was fought 2295 yrs. $\left\{\begin{matrix}\text{before Waterloo.}\\ \text{after Salamis.}\end{matrix}\right.$

So, it $\left\{\begin{matrix}\text{was}\\ \text{is}\end{matrix}\right.$ 30° $\left\{\begin{matrix}\text{colder yesterday}\\ \text{warmer to-day}\end{matrix}\right.$ than $\left\{\begin{matrix}\text{to-day.}\\ \text{yesterday.}\end{matrix}\right.$

In such double statements, all the words and signs in the upper line, together with the common parts, go to make up the first statement; and all the words and signs in the lower line, together with the common parts, the second statement. In the same way, three or more statements may be written together.

When, of a double statement, only one part can be true, but which that is, is unknown, such statement is *ambiguous*.

§ 5. COPULAS AND STATEMENTS.

Two numbers are *equal* when, in every combination which contains either of them, the other may take its place without changing the result.

When one number is equal to another, the two are joined by the sign $=$, read *equals*, or *is equal to*, and the whole is an *equation;* or by the sign $\equiv$, read *is identical with*, and the whole is an *identity*.

E.g., 100 cents $=$ 1 dollar; 100 cents $\equiv$ 100 cents; $x \equiv x$.

An identity is an equation wherein the two numbers remain equal, however the values of any of the letters may change. Every identity is an equation, but not every equation is an identity. Hence $=$ may always take the place of $\equiv$, but $\equiv$ not always of $=$.

The sign $\equiv$ is also used for "stands for" and "represents."

E.g., $p \equiv$ principal, $t \equiv$ time, $r \equiv$ rate, $i \equiv$ interest.

When one number is not equal to another, they are joined by the signs $\neq$, $\not\equiv$, $<$, $>$, $\lessdot$, $\gtrdot$, read: *not equal to, not identical with, less than, greater than, smaller than, larger than.*

E.g., 80 cts. $\dot{\neq}$ \$1, 100 cts. $\not\equiv$ \$1, 80 cts. $<$ \$1, 120 cts. $>$ \$1,
80 cts. $\lessdot$ \$1, 120 cts. $\gtrdot$ \$1.

So, $\not<$, $\not>$, $\not\lessdot$, $\not\gtrdot$, mean *not less than*, *not greater than*, etc.

The words "greater" and "less" are here used in a technical sense, and may be expressed by *higher* and *lower* in speaking of temperatures and elevations, by *north of* or *east of* and *south of* or *west of* in Surveying and Geography, by *later* and *earlier* in comparing two dates, and so on; but "larger" and "smaller" take account of the size of the two numbers only.

E.g., 30 ft. up $>$ 50 ft. down, and 30 ft. down $>$ 50 ft. down;
i.e., $^{+}30 > {}^{-}50$, and $^{-}30 > {}^{-}50$.

But $^{+}30 \lessdot {}^{-}50$, and $^{-}30 \lessdot {}^{-}50$.

If two numbers be equally large, the sign is Ξ; if not equally large, $\not\Xi$.

E.g., $^{+}1600 \;\Xi\; {}^{-}1600$; $^{+}1600 \not\Xi {}^{-}1700$.

In general, any positive number, however small, is greater than any negative number, however large; and, of two negative numbers, the smaller is greater than the larger.

Manifestly, the greater a number the less is its opposite; but a number and its opposite are equally large.

The signs $=$, $\equiv$, $\equiv$, $\neq$, $\not\equiv$, $\not\equiv$, $<$, $>$, $\lessdot$, $\gtrdot$, $\not<$, $\not>$, $\not\lessdot$, $\not\gtrdot$ are *signs of assertion* or *copulas*.

Equations, identities, and inequalities are *statements*, and when of general truths, they are *formulae*. The *first member* is all that precedes the copula, and the *second member*, all that follows it.

A *continued statement* is one having more than two members; it is equivalent to as many simple statements as there are copulas, and each copula, unless preceded by a comma, connects the two members immediately adjacent to it.

E.g., $1 < 3 < 5 \lessdot {}^{-}7 \not\gtrdot 9$

is equivalent to the group of independent statements

$$1 < 3,\ 3 < 5,\ 5 \lessdot {}^{-}7,\ {}^{-}7 \not\gtrdot 9.$$

So, $\because a \lessdot b, \quad \therefore 2a \lessdot 2b \lessdot 3b$

is equivalent to the chain of connected statements

$$\because a \lessdot b, \quad \therefore 2a \lessdot 2b, \text{ and } 2b \lessdot 3b, \text{ and } \therefore 2a \lessdot 3b.$$

But $\because a \lessdot b, \quad \therefore a, \lessdot 2a, \lessdot 2b$

is equivalent to the chain of connected statements

$$\because a \lessdot b, \quad \therefore a \text{ (which } \lessdot 2a) \lessdot 2b,$$

and is read: Since a is smaller than b, therefore a, which is smaller than $2a$, is smaller than $2b$.

This is, in effect, a brief form for a logical chain of statements. The office of the commas is to parenthesize what is between them, and compare directly what precedes the first comma and what follows the last comma; the basis of comparison being found in what the commas enclose. The first comma is read *which*.

So, $a \neq -a, = b$,

is equivalent to the two independent statements

$$a \neq -a \quad \text{and} \quad a = b;$$

and here, too, the office of the comma is to carry forward the first member, a, and compare it with b which follows the comma. The comma is read *and*.

§ 6. ADDITION.

THE SUM of two or more concrete numbers of the same kind is a new concrete number got by joining together the several things measured, and then measuring the aggregate by the same unit that measured the original numbers.

The sum of two or more abstract numbers is a new abstract number which, if used as an operator upon any unit, will give the same result as if the original numbers were first used as operators upon the unit and their results were then added.

Addition is the process of finding the sum of two or more numbers. If the numbers added be commensurable, then, at bottom, addition is but counting either by entire units or by the aliquot parts of a unit: on (forward) if positive numbers be added; off (backward) if negative numbers be added.

The sign of addition is $+$; read *plus*, or the *sum of ... and ...*

E.g., 50 cts. + 60 cts. + 90 cts. = \$2; $50 + 60 + 90 = 200$.

In Algebra the word "addition" is used in a broader sense than in Arithmetic, and covers negative as well as positive numbers.

E.g., he who has \$10,000 cash and \$4,000 debts is worth but \$6,000;

i.e., \$10,000 cash + \$4,000 debts = \$6,000 net assets;
$^{+}10,000 + {}^{-}4,000 = {}^{+}6,000$.

So, a train which has run east 10 miles, then west 20 miles over the same track, is 10 miles west of the starting-point;

i.e., 10 east-miles + 20 west-miles = 10 west-miles;
$^{+}10 + {}^{-}20 = {}^{-}10$.

But a train which has run west 10 miles, then west 20 miles more, is 30 miles west of the starting-point;

i.e., 10 west-miles + 20 west-miles = 30 west-miles;
$^{-}10 + {}^{-}20 = {}^{-}30$.

Though the numbers to be added must always be of the same kind, they are often expressed by letters whose values are not known, or in units whose values are different, and which therefore cannot be reduced to one sum.

E.g., $5^h 33^m 30^s + 12^h 47^m 30^s = 18^h 21^m$.

Manifestly, the sum of two opposites is 0.

E.g., 90 ft. up is the opposite of 90 ft. down;

i.e., $+90$ is the opposite of -90; and the sum of the two is 0.

So, $+a$ and $-a$, ^{-}a and ^{+}a, $2b-3c$ and $3c-2b$.

§ 7. SUBTRACTION.

SUBTRACTION is the inverse of addition, and consists in finding what number must be added to one number, the *subtrahend*, to give another number, the *minuend*. The result is the *remainder*, and the sign is $-$, read *minus* or the *excess of* ... *over* ... One or both of the numbers may be negative, and the minuend may be less than the subtrahend.

E.g., $\$50 - \$40 = \$10$, $\$40 - \$50 = {}^{-}\$10$.

$^{-}\$50 - {}^{-}\$40 = {}^{-}\$10$, $^{-}\$40 - {}^{-}\$50 = {}^{+}\$10$.

So, if of two men A has $10,000 cash and no debts, and B has $5,000 debts but no assets,

then A is $15,000 better off than B,

i.e., $^{+}10{,}000 - {}^{-}5{,}000 = {}^{+}15{,}000$;

and B is $15,000 worse off than A,

i.e., $^{-}5{,}000 - {}^{+}10{,}000 = {}^{-}15{,}000$.

So, $\because$ the battle of Salamis was fought 480 B.C., and that of Waterloo 1815 A.D.,

$\therefore$ Waterloo was fought 2295 years after Salamis,

i.e., $^{+}1815 - {}^{-}480 = {}^{+}2295$;

and Salamis was fought 2295 years before Waterloo,

i.e., $^{-}480 - {}^{+}1815 = {}^{-}2295$.

So, if to-day a thermometer read 10° below zero, and yesterday it read 20° below zero,

then it is 10° warmer to-day than yesterday,

i.e., $^{-}10 - {}^{-}20 = {}^{+}10$;

and it was 10° colder yesterday than to-day,

i.e., $^{-}20 - {}^{-}10 = {}^{-}10$.

The *difference* between two numbers is the remainder found by subtracting the less from the greater; the sign is $\sim$.

E.g., $16 \sim 12 = 12 \sim 16 = 4$; $^{-}16 \sim {}^{+}12 = {}^{+}16 \sim {}^{-}12 = 28$.

§ 8. MULTIPLICATION.

THE PRODUCT of a concrete number, the *multiplicand*, by an abstract number, the *multiplier*, is a concrete number of the same kind as the multiplicand, and bearing to the multiplicand the same relation as the multiplier bears to unity.

The product of two or more abstract numbers is a new abstract number such that, if a unit be multiplied by it, the product is the same as the final product obtained by multiplying the unit by the first of the numbers, the product so found by the second of them, and so on.

Multiplication is the process of finding the product of two or more numbers; the numbers are the *factors* of the product.

Multiplication by a $\left\{\begin{matrix}\text{positve}\\ \text{negative}\end{matrix}\right.$ integer is but a repeated $\left\{\begin{matrix}\text{addition}\\ \text{subtraction}\end{matrix}\right.$ of the multiplicand $\left\{\begin{matrix}\text{to}\\ \text{from}\end{matrix}\right.$ 0, and multiplication by a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ fraction is the repeated $\left\{\begin{matrix}\text{addition to}\\ \text{subtraction from}\end{matrix}\right.$ 0 of the equal parts into which the multiplicand is divided.

In the last analysis, multiplication is but a counting, on or off, according as the multiplier is positive or negative; but it is a counting by groups, each equal to the multiplicand, if the multiplier be an integer, and by aliquot parts of such groups if it be a fraction, instead of by single units as in addition.

E.g., five, ten, fifteen, twenty, twenty-five, thirty, gives the product of five by six, or of $^-$five by $^-$six.

So, $^-$five, $^-$ten, $^-$fifteen, gives the product of $^-$five by three, or of five by $^-$three.

So, one half of five, two halves of five, three halves of five, gives the product of five by $\frac{3}{2}$, or of $^-$five by $^-\frac{3}{2}$.

So, $\frac{1}{2}$ of $\frac{5}{7}$, $\frac{2}{2}$ of $\frac{5}{7}$, $\frac{3}{2}$ of $\frac{5}{7}$, gives the product of $\frac{5}{7}$ by $\frac{3}{2}$, or of $^-\frac{5}{7}$ by $^-\frac{3}{2}$.

So, $\frac{3}{4}$, $\frac{6}{4}$, $\frac{9}{4}$ gives the product of $\frac{3}{4}$ by 3 or of $^-\frac{3}{4}$ by $^-3$.

The signs of multiplication are $\times$, read *by*, and $\cdot$, read *into*.

E.g., 50 cts. $\times 8 =$ \$4 ; $8 \cdot 50$ cts. $=$ \$4.

So, placing the factors one after the other, with no sign between them, means multiplication of the first into the second, or of the second by the first.

E.g., ab is the product of a into b, or of b by a,

and $ab \equiv a \cdot b \equiv b \times a$.

When the product of two numbers is multiplied by a third number, such multiplication is the *continued multiplication* of the three numbers; so for four numbers, for five numbers, and so on; and the product of such multiplication is the *continued product* of the several factors.

E.g., $5 \times 6 \times 7 = 210$, and $5 \cdot 6 \cdot 7 = 210$.

The continued product of the natural numbers $1 \cdot 2 \cdot 3 \ldots$ is indicated by the sign ! placed after the last factor, or by the sign $\lfloor$ placed before and under the last factor.

E.g., $5!$ or $\lfloor\underline{5}$, read *factorial* 5, $\equiv 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, $= 120$;

$n!$ or $\lfloor\underline{n}$, read *factorial* n, $\equiv 1 \cdot 2 \cdot 3 \cdot \ldots n$.

Some peculiar properties of negatives appear in multiplication.

E.g., if a train, now at A, is running east 20 miles an hour,

then five hours hence it will be 100 miles east of A,

i.e., $^{+}20 \times {}^{+}5 = {}^{+}100$;

but five hours ago it was 100 miles west of A,

i.e., $^{+}20 \times {}^{-}5 = {}^{-}100$.

So, if the train is backing, *i.e.* running west,

then five hours hence it will be 100 miles west of A,

i.e., $^{-}20 \times {}^{+}5 = {}^{-}100$;

but five hours ago it was 100 miles east of A,

i.e., $^{-}20 \times {}^{-}5 = {}^{+}100$.

Two numbers whose product is 1 are *reciprocals* of each other.

E.g., 4 is the reciprocal of $\frac{1}{4}$; -3 of $-\frac{1}{3}$; $\frac{7}{3}$ of $\frac{3}{7}$.

Manifestly, the larger a number, the smaller its reciprocal.

The product of a number by an integer is a multiple of that number: the double, triple, quadruple, $\cdots$, when the multiplier is 2, 3, 4, $\cdots$.

§ 9. DIVISION.

Division is the inverse of multiplication, and consists in finding either factor, when the product and the other factor are given. The product is now called the *dividend*, the given factor is the *divisor*, and the result is the *quotient*.

E.g., $\because$ the product of 5 by 10 is 50,

$\therefore$ the quotient of 50 by $\begin{cases} 5 \text{ is } 10. \\ 10 \text{ is } 5. \end{cases}$

So, $\because$ the product of a into b is ab,

$\therefore$ the quotient of ab by $\begin{cases} b \text{ is } a. \\ a \text{ is } b. \end{cases}$

The $\begin{cases} \text{quotient,} \\ \text{divisor,} \end{cases}$ being the multiplier, is an abstract number [§ 8]; and the $\begin{cases} \text{divisor} \\ \text{quotient} \end{cases}$ and dividend are alike in kind. When both factors are abstract, the two definitions of division agree, as will appear later.

E.g., $\because$ the product of \$5 by 4 is \$20,

$\therefore$ the quotient of \$20 by $\begin{cases} \$5 \text{ is } 4. \\ 4 \text{ is } \$5. \end{cases}$

The signs of division are :, read *the ratio of ... to ...*, and $\div$, read *divided by*, or *the quotient of ... by ....*

E.g., $\$20 : \$5 = 4$; $\$20 \div 4 = \$5$; $20 : 5 = 4$; $20 \div 4 = 5$.

So, writing the dividend over the divisor with a horizontal line between them means division. The dividend is then called the *numerator*, the divisor the *denominator*, and the whole expression a *fraction*. Hence a fraction is the expression for the quotient in a division as yet unperformed.

Note. This definition of a fraction differs from that heretofore given [§ 1], but later it will appear that the two definitions are in full accord.

If the dividend be a multiple of the divisor, then the quotient is an integer and the division is complete; but if the dividend be not a multiple of the divisor, its excess over the greatest multiple that is contained in it is the *remainder*.

E.g., $27 : 5 = 5$, quotient with 2 remainder.

§ 10. INVOLUTION.

A $\left\{ \begin{matrix} \textit{positive} \\ \textit{negative} \end{matrix} \right.$ *integral power* of a number is the continued $\left\{ \begin{matrix} \text{product} \\ \text{quotient} \end{matrix} \right.$ of unity by the given number.

The number whose power is sought is the *base.*

The symbol that shows how many times the base is used as $\left\{ \begin{matrix} \text{multiplier} \\ \text{divisor} \end{matrix} \right.$ is the *exponent;* it is written at the right and above the base, and is $\left\{ \begin{matrix} \text{positive} \\ \text{negative} \end{matrix} \right.$ for a $\left\{ \begin{matrix} \text{positive} \\ \text{negative} \end{matrix} \right.$ power.

E.g., $1 \times a \times a \times a = a^3$, read
third power of a, a *third power*, or a *cube.*
$1 \times a \times a = a^2$, read
second power of a, a *second power*, or a *square.*
$1 \times a = a^1$, read
first power of a, a *first power*, or simply a.
$1 = a^0$, read
zeroth power of a, a *zeroth power.*
$1 \div a = a^{-1}$, read
minus first power of a, or a *minus first power.*
$1 \div a \div a = a^{-2}$, read
minus second power of a, or a *minus second power.*
$1 \div a \div a \div a = a^{-3}$, read
minus third power of a, or a *minus third power.*

A *root* of a number is one of the equal factors into which it may be resolved. The number whose root is sought is the base; the symbol that shows into how many equal factors the base is resolved is the *root-index.* The *radical sign*, $\sqrt{\ }$, is written before the base, and the root-index is at the left and above it; or else the reciprocal of the root-index is attached to the base as an exponent. The root-index 2 need not be written.

E.g., $\sqrt[2]{4}$, or simply $\sqrt{4} = 4^{\frac{1}{2}} = {}^{\pm}2$; $\sqrt[3]{243} = 243^{\frac{1}{3}} = 7$.

A *fractional power* of a number is either a root of the number or some integral power of such root. The exponent is then a simple fraction whose denominator shows into how many equal

factors the base is resolved, and whose numerator shows how many times one of these factors is used as $\left\{\begin{array}{l}\text{multiplier.}\\ \text{divisor.}\end{array}\right.$

E.g., $64^{\frac{2}{3}} = 1 \times 4 \times 4 = 16$, read 64, $\frac{2}{3}$ *power, equals* 16.

$64^{-\frac{2}{3}} = 1 \div 4 \div 4 = \frac{1}{16}$, read 64, $-\frac{2}{3}$ *power, equals* $\frac{1}{16}$.

So, $a^{\frac{1}{2}} \equiv \sqrt{a}$, $c^{\frac{2}{3}} \equiv (\sqrt[3]{c})^2$, $x^{\frac{m}{n}} \equiv (\sqrt[n]{x})^m$, $k^{-\frac{p}{q}} \equiv (\sqrt[q]{k})^{-p}$.

The words "integral," "fractional," "positive," and "negative" apply to the exponents only, and not at all to the results of the operations indicated; *i.e.*, a positive integral power is one whose exponent is a positive integer, and so on.

Integral and fractional powers are *commensurable powers.* Those powers whose exponents are incommensurable are called *incommensurable powers;* they are defined in [VIII. § 4].

Involution is the process of finding the powers of numbers; its sign is the position of the exponent.

NOTE. — The reader may compare what is here said of positive and negative exponents, as indices of repeated multiplication and division of a unit by the base, or by one of the equal factors of the base, with what is said in § 3 of operations which tend to neutralize each other. He will then see the peculiar propriety of expressing repeated multiplication by a positive exponent, and repeated division, the inverse of multiplication, by a negative exponent.

§ 11. EVOLUTION.

EVOLUTION is the inverse of involution, and consists in finding a base that, when raised to the power denoted by the index, produces the given number. The result is the root.

The *logarithm* of a number is the exponent of that power to which a base must be raised to give the number. The *finding of logarithms* is another inverse of involution.

E.g., $\because 10^2 = 100$, $\therefore$ 2 is the logarithm of 100 taken to the base 10; it is written $\log_{10} 100 = 2$, and read *log, base* 10, *of* 100 *equals* 2.

So, $\log_{10} 1000 = 3$, $\log_{10} 10 = 1$, $\log_{10} 1 = 0$, $\log_{10} .1 = -1$.

§ 12. EXPRESSIONS.

AN ALGEBRAIC EXPRESSION is a number or combination of numbers written in algebraic form. It is called an "expression" or a "number," according as the thought is of the symbol or of the value which the symbol represents.

Unless a single letter or numeral, an expression is made up of simpler expressions affected or combined by signs of operation; and the order of these operations is as follows:

1. Every letter or numeral, with its indices, if any, denotes a number by itself; and so does every expression united by a bar or parenthesis. These numbers, in turn, may be affected by exponents, etc.; but each exponent affects only the single numeral, letter, or parenthesis it is written to; and if a power of a power is to be denoted, the new base must be parenthesized.

E.g., $2^3 3^2 a^3 b^4 (x-y)(x+y)^2$ is the product of 2^3, 3^2, a^3, b^4, $(x-y)$, and $(x+y)^2$;

but $[(2^3 3)^2 a]^3$ is the cube of the product of a by the square of $2^3 3$.

So, $(a^b)^c$ is the cth power of a^b;

but a^{b^c} is the b^cth power of a.

So, $a^{b^{c^d}}$ is the b^{c^d}th power of a.

2. When a product is denoted by writing the factors together without the sign $\times$ or $\cdot$, or when a quotient is denoted by a fraction, the product or quotient is affected, as a single number, by the adjacent sign $\sqrt{}$, log, $\times$, $\cdot$, $:$, $\div$, $+$, or $-$.

E.g., $\sqrt{}2ab \cdot x^2y^3 : 3^4$ denotes that the square root of $2ab$ is multiplied into x^2y^3, and the product divided by 3^4;

but $\sqrt{2ab \cdot x^2y^3 : 3^4}$ is the square root of $2ab \cdot x^2y^3 : 3^4$.

So, $\log \frac{2}{3}y$ is the logarithm of $\frac{2}{3}y$;

but $\log \frac{2}{3} \cdot y$ is the product of $\log \frac{2}{3}$ into y.

3. When successive numbers are separated, some by the sign $\times$, $\cdot$, $:$, or $\div$, and some by $+$ or $-$, the multiplications and divisions are first performed, and then the products and quotients are added or subtracted. If several of these signs of

multiplication and division occur in succession, or several signs of addition and subtraction, the left-hand operation is first performed.

E.g., $3:2\cdot 6-6\div 3\times 2+1$ denotes that from $\frac{3}{2}\cdot 6$, or 9, is subtracted $\frac{6}{3}\times 2$, or 4, and to the remainder, 5, is added 1.

Those parts of an expression which are joined by the signs + or − are *terms*, and terms are $\left\{\begin{matrix}\textit{complex}\\ \textit{simple}\end{matrix}\right.$ which $\left\{\begin{matrix}\text{contain}\\ \text{do not contain}\end{matrix}\right.$ the sign + or − except in an index.

An expression of one term only is a *monomial*, of two terms a *binomial*, of three terms a *trinomial*, of four terms a *quadrinomial;* of two or more terms a *polynomial.*

An expression is $\left\{\begin{matrix}\textit{numerical}\\ \textit{literal}\end{matrix}\right.$ when the numbers are expressed $\left\{\begin{matrix}\text{wholly by numerals;}\\ \text{wholly or in part by letters;}\end{matrix}\right.$ $\left\{\begin{matrix}\textit{finite}\\ \textit{infinite}\end{matrix}\right.$ when the number of operations implied is $\left\{\begin{matrix}\text{limited.}\\ \text{unlimited.}\end{matrix}\right.$

A finite expression is $\left\{\begin{matrix}\textit{algebraic}\\ \textit{transcendental}\end{matrix}\right.$ when there is implied $\left\{\begin{matrix}\text{no}\\ \text{some}\end{matrix}\right.$ other operation than addition, subtraction, multiplication, division, and involution to commensurable powers.

An algebraic expression is $\left\{\begin{matrix}\textit{rational}\\ \textit{surd}\end{matrix}\right.$ when it $\left\{\begin{matrix}\text{can}\\ \text{cannot}\end{matrix}\right.$ be freed from roots; $\left\{\begin{matrix}\textit{entire}\\ \textit{fractional}\end{matrix}\right.$ when $\left\{\begin{matrix}\text{free from divisors and roots.}\\ \text{not free from divisors.}\end{matrix}\right.$

E.g., $3bc$, A, $\text{A}''_{r+1}x^{r+1}$ [r being a positive integer] are entire simple rational monomials.

So, $\frac{3+2x}{a}$, $\frac{3y}{a+1}$, $\frac{a+1}{a-1}$, $(a+x^{-1})$ are complex fractional monomials.

$a+x^{-1}$ is a fractional binomial with simple terms.

$(a+b)+\frac{mn}{m+n}$ is a binomial with complex terms.

$3bc+5xy-7mn$ and $\frac{2x}{a}+\frac{3y}{a+1}-na^{n-1}x$ are rational trinomials; the first is entire, the second fractional.

So, $\sqrt{a} \pm 1 \mp 1 - \sqrt{a}$ and $5pq^{\frac{2}{3}} - 3pq^{\frac{2}{3}} + pq^{\frac{2}{3}} - \frac{1}{2}pq^{\frac{2}{3}}$ are quadrinomials, but reducible to monomials; *viz.*, to 0 and $2\frac{1}{2}pq^{\frac{2}{3}}$.

The above examples are literal; the following are numerical:

$\sqrt{5} + \sqrt{7}$, $2^{\frac{2}{3}} + 3^{\frac{2}{3}}$, $1 + \sqrt{-1}$ are binomial surds.

$1 + \sqrt[3]{2} - \sqrt[7]{-3}$ is a trinomial surd.

$1.1 \times 1.01 \times 1.001 \times \cdots$ is an infinitely continued product.

$$\cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}$$

is an infinite continued fraction, but one whose value is $\sqrt{2} - 1$, an irrational finite number, as will appear later.

An expression may be entire, rational, etc., as to some of its letters only.

E.g., $\dfrac{a + \sqrt{bc}}{m + n}$ is rational as to a, m and n,

and it is entire as to a;

but it is irrational as to b and c,

and it is fractional as to m and n.

When the terms of an expression are so related to each other that each successive term is derivable by some fixed law from the previous terms, the expression is a *series*.

E.g., $1 + x + x^2 + x^3 + \cdots + x^r$ is a finite series if r is any given integer;

but $1 + x + x^2 + x^3 + \cdots + x^r + \cdots$ is an infinite series.

In this series x^r is called the *general* term, because by giving to r in turn the values 0, 1, 2, 3, ..., or any of them, all the terms of the series, or any of them, are found.

When the values of the several letters in a literal expression are known, then the value of the expression may be found by *substituting* these values in place of the letters, and performing the operations indicated.

E.g., if $a = 2$, $b = 3$, $c = 4$,

then $abc = 24$, $a + b - c = 1$, $a : (b + c) = \frac{2}{7}$.

So, if $x = a + b$ and $y = a - b$,

then $x + y = 2a$, $x - y = 2b$, $xy = a^2 - b^2$.

A literal expression may be entire, fractional, rational, etc., but its numerical value not so; or the reverse.

E.g., x is entire, x^{-1} fractional, $\sqrt{x}$ irrational, y^2 rational;
but, if $x = \frac{1}{4}$ and $y = \sqrt[3]{2}$,
then the value of x is fractional, that of x^{-1} is entire, that of $\sqrt{x}$ is rational, and that of y^2 is irrational.

Manifestly, if all the letters stand for integers and the expression is entire, its value is an integer.

E.g., if a and b are integers, $(a-b)(2a^2+3b^3)$ is integral.

As to any of its letters, an expression is *symmetric* when its value remains unchanged however those letters exchange places.

E.g., xyz and $x+y+z$ are symmetric as to x, y, and z, or as to any two of them.
So, $w+x-y-z$ is symmetric as to w and x, and as to y and z; but not as to w and y, to w and z, to x and y, nor to x and z.

An expression is *converted* or *transformed*, when changed in form but not in value; *developed* or *expanded*, when transformed into a series.

§ 13. FUNCTIONS.

If a number is so related to other numbers that its value depends upon their values, it is a *function* of those numbers: an $\left\{\begin{matrix}\textit{explicit}\\ \textit{implicit}\end{matrix}\right.$ function, when $\left\{\begin{matrix}\text{expressed}\\ \text{not expressed}\end{matrix}\right.$ in terms of those numbers. The numbers are the *arguments* of the function.

E.g., in $u = 3xy$, u is an explicit function of the arguments x and y;
but x is an implicit function of the arguments u and y,
and y is an implicit function of the arguments u and x.
So, in $y^2 = u : 3x$, y is an implicit function of u and x;
but in $y = \sqrt{(u : 3x)}$, y is an explicit function.

An explicit function of one or more numbers is *known* (given or determined) in terms of those numbers. It is *symmetric*, *algebraic*, *transcendental*, *rational*, etc., according as the expression which gives its value is symmetric, algebraic, etc.

If one number (function) depends upon its arguments in the same way as another number depends upon its own arguments, *i.e.*, if the expressions involved are of the same form, then the first number is the *same function* of its arguments, as the second number is of its arguments.

E.g., if $x^2+x=a$ and $y^2+y=b$,

then $\left\{\begin{matrix} a \\ x \end{matrix}\right.$ is the same $\left\{\begin{matrix} \text{explicit} \\ \text{implicit} \end{matrix}\right.$ function of $\left\{\begin{matrix} x \\ a \end{matrix}\right.$ as $\left\{\begin{matrix} b \\ y \end{matrix}\right.$ is of $\left\{\begin{matrix} y. \\ b. \end{matrix}\right.$

So, the expression $x+2y$ is the same function of x and y as $a+2b$ is of a and b, and the same as $y+2x$ is of y and x.

A function may be denoted by the letters f, F, ϕ, $\cdots$, with or without indices, and followed by the arguments enclosed in a parenthesis.

To denote $\left\{\begin{matrix} \text{the same} \\ \text{a different} \end{matrix}\right.$ function $\left\{\begin{matrix} \text{the same letter} \\ \text{a different letter or index} \end{matrix}\right.$ is used.

E.g., if $f(x) \equiv x^2 - ax$,

then $f(y) \equiv y^2 - ay$ during the same investigation;

but $f(y)$ cannot stand for $a^2 - ay$, nor for $ay - y^2$.

So, if $\text{F}(x, a) \equiv x^2 - ax$,

then $\text{F}(y, b) \equiv y^2 - by$, $\text{F}(a, x) \equiv a^2 - xa$, $\cdots\cdot$

But if $\text{F}'(x, a) \equiv x^3 - a^2x$, or any other form,

then $\text{F}'(b, y) \equiv b^3 - y^2 b$, the same form.

If $\text{F}(x, y) \begin{matrix} \equiv \\ \not\equiv \end{matrix} \text{F}(y, x)$,

then either is $\left\{\begin{matrix} \text{a symmetric} \\ \text{an unsymmetric} \end{matrix}\right.$ function of x and y.

E.g., if $\text{F}(x, y) \equiv f(x) \cdot f(y)$, or $\equiv f(xy)$,

then F denotes a symmetric function;

but not if $\text{F}(x, y) \equiv f(x) \cdot f'(y)$, or $\equiv f(x : y)$.

So, $\phi(x, y) + \phi(y, x)$, but not $\phi(x, y) - \phi(y, x)$, is symmetric.

So, if $\text{F}(x, y, z)$, $\text{F}(x, z, y)$, $\text{F}(y, z, x)$, $\text{F}(y, x, z)$, $\text{F}(z, x, y)$, and $\text{F}(z, y, x)$ be $\left\{\begin{matrix} \text{all} \\ \text{not all} \end{matrix}\right.$ identical,

then either is $\left\{\begin{matrix} \text{a symmetric} \\ \text{an unsymmetric} \end{matrix}\right.$ function of x, y, z.

§ 14. COEFFICIENTS. — LIKE AND UNLIKE TERMS.

WHEN a number is the product of several factors, they are its *co-factors;* and any one of them, or the product of any two or more of them, is a *coefficient* of the product of the remaining co-factors. A coefficient is *numerical*, *literal*, or *mixed*, according as it is a numeral, a letter or letters, or a numeral and letters combined.

E.g., in $7abc$, 7 is the coefficient of abc, $7a$ of bc, $7ab$ of c, $7b$ of ac, $7c$ of ab, $7bc$ of a, ...

Usually the numeral alone, together with the sign of the number, $+$ or $-$, is counted as the coefficient.

Terms which differ only in their coefficients are *like* (similar) terms; other terms are unlike.

E.g., $5ax$ and $7ax$ are like, but $5ax$ and $7by$ are unlike.

So, $5ax$ and $7bx$ are like if $5a$ and $7b$ are counted as the coefficients of x; but unlike if 5 and 7 be coefficients of ax and bx.

So, $3\sqrt{(a^2+b^2)}$, $5a\sqrt{(a^2+b^2)}$, $(7b+9c)\sqrt{(a^2+b^2)}$ are like surds.

But $3\sqrt{(a^2+b^2)}$, $5a\sqrt[3]{(a^2+b^2)}$, $(7b+9c)\sqrt{(a^2+c^2)}$ are all unlike surds.

§ 15. DEGREE.

THE sum of the exponents in a simple term is its *degree.* The degree of a polynomial is that of the term whose degree is highest of all. A polynomial made up of simple terms all of the same degree is *homogeneous.* Expressions having the same degree are homogeneous with each other.

E.g., $a^3+3a^2b+3ab^2+b^3$ is homogeneous, of the 3d degree.

So, a^n, $a^{n-1}b$, $a^{n-2}b^2$, ..., $a^{n-r}b^r$, ..., ab^{n-1}, b^n are homogeneous with each other and of the nth degree.

and ax^2, b^2xy, c^3y^2 are of the 2d degree and homogeneous with each other as to x and y;

but of the 3d, 4th, and 5th degrees respectively, and not homogeneous, as to all the letters.

So, the trinomial $a^{-1}+b^{-2}+c^{-3}$ is of the -1st degree, and not homogeneous;

and $a^1+b^2+c^3$ is of the 3d degree and not homogeneous;

and $ma^{-4}b^{-1}x^2+n^2a^3b^{-8}xy+p^3a^4b^{-9}y^2$ is homogeneous and of the 2d degree as to x and y, and homogeneous and of the -3d degree as to a, b, x, and y;

but it is not homogeneous if m, n, and p be also counted, for then the first term is of the -2d degree, the second term is of the -1st degree, and the last term is of the 0th degree.

So, the binomials $\frac{a^6}{b^5}-\frac{3b^4}{2a^3}$ and $\sqrt[3]{a^2}-\left(3\sqrt[3]{b^4}-\sqrt[3]{\frac{a^6}{b^2}}\right)b^{-\frac{2}{3}}$ are homogeneous and respectively of the 1st and the $\frac{2}{3}$ degree.

The degree of a product is the sum of the degrees of the factors. The degree of any power of an expression is the product of the degree of the expression by the exponent of the power. A power or product of homogeneous expressions is homogeneous, and a sum of homogeneous expressions of any same degree is homogeneous and of that degree.

Thus, [illegible] is of the 1st degree and homogeneous as to all the letters.

but [illegible] is of the 2d degree and not homogeneous as to a and b only, or as to x and y only.

§ 16. EXAMPLES.

1. In the sentence $(x+a)^2-(x-a)^2=4ax$, point out the verb, nouns, conjunctions, and phrases, and state their grammatical relations.

§ 2.

2. Translate and read in words the following symbols:

   3^2, a'_0, b''_1, $c^{(n)}_r$, $d_{(n),x}$, $\mathrm{N}^{(n')}$, $\mathrm{D}_{x'}$, $\mathrm{F}''_{y''}$, $x^{\mathrm{v}}_{n,q'''}$.

3. Write in symbols:

   p sub naught, q second, x prime sub r prime, large x fourth sub a prime, large f sub i third and sub k.

§ 3.

4. If $a=2$ and $b=-3$, which of the following numbers are positive, and which negative?

   a, b, $-a$, $-b$, $2a$, $5b$, $-8a$, $-11b$.

§ 5.

5. Connect each of the following pairs of numbers by the appropriate sign $>$ or $<$; also by the sign $\ngtr$ or $\nless$:

   $0,1$; $0,-1$; $-1,0$; $2,1$; $1,-2$; $-2,-1$; $-1,-2$; ${}^-x, {}^-2x$; ${}^+x, {}^-2x$; ${}^-a+{}^-b, {}^-a-{}^-b$.

6. Read in words the statements:

   If $a \nless b \stackrel{<}{>} 0$, then $a \stackrel{>}{<} b$;

   If $a \ngtr b, \stackrel{<}{>} 0$, then $a \stackrel{<}{>} b$;

   and explain the meaning of the copulas used therein.

7. Correct the following continued statements by introducing or suppressing commas:

   $3 \nless -4, < 1 \ngtr 1$; $\because x \ngtr a$, $\therefore 3x \ngtr 3a, \ngtr 2a$.

§§ 6, 7.

8. Read in words the following formulæ:

   ${}^-a+{}^-b={}^-({}^+a+{}^+b)$; ${}^+a-{}^-b>0$; ${}^-a-{}^+b<0$.

9. Correct the following statements by introducing the proper brackets:

   $5-3+1=1$; $5-3-1=3$; $-5+4-1=-8$.

10. Read in words the statements:
$(a+b)+(a-b)=2a$; $(a+b)-(a-b)=2b$;
and, considering a and b to stand for any two numbers whatever, read these two statements as general truths.

§§ 8, 9.

11. Separate the portions of the following continued statements where necessary to avoid false equations or inequalities:
$2 \times 3 = 6 + 4 = 10 \div 5 = 2$;
$2xy + 3xy = 5xy - xy = 4xy$.

12. Read and verify the statements:
$1! \cdot 2! \cdot 3! \cdot 4! = 1^4 \cdot 2^3 \cdot 3^2 \cdot 4^1$; $3!!, \equiv (3!)!, = (3!)^2 \cdot 4 \cdot 5$.

13. Correct the following statements by introducing the proper brackets:
$30 \div 3 \times 5 = 2$; $30 \div 10 \div 5 = 15$; $5x - 4x \times 1 + 2 = 3x$.

§ 10.

14. Translate into words:
$$(a+b)^3 + (x-y)^2;\ \begin{array}{r|r|l} x^3 + a & x^2 + ab & x + abc; \\ +b & +bc & \\ +c & +ca & \end{array}$$
$$[(a+b) \cdot c - (x-y)^2]^n \cdot \{[(a-b)-c]^q - (x-y)^p\}^{-3}.$$

15. Interpret the following expressions and statements:
$2^{\frac{1}{2}}$; $a^{\frac{2}{3}}$; $(2x)^{-3}$; $(6y)^{-\frac{3}{2}}$; $8^{\frac{2}{3}} = 4$; $2^0 = 3^0 = 4^0 = \cdots$.

16. Introduce brackets so that 2^{2^3} shall equal 64; 256.

17. What power of x is $[(x^3)^2]^2$? $(x^3)^{2^2}$? $(x^{3^2})^2$? $x^{(3^2)^2}$? $x^{3^{2^2}}$?

§ 11.

18. Find the value of:
$\log_2 8$, $\log_2 \frac{1}{8}$, $\log_2 2$, $\log_2 1$, $\log_2 2\sqrt[3]{2}$, $\log_{\frac{1}{2}} 8$,
$\log_{\frac{1}{2}} \frac{1}{8}$, $\log_8 2$, $\log_8 \frac{1}{2}$, $\log_{\frac{1}{8}} 2$, $\log_{\frac{1}{8}} \frac{1}{2}$, $\log_4 16$.

19. Of what number is 4 the logarithm to base 2? to base 4? to base $\frac{1}{2}$? to base $\frac{1}{8}$?

20. To the base 10, of what number is 3 the logarithm? 2? 1? 0? -1? -2? $\frac{3}{2}$? $\frac{1}{2}$? $-\frac{1}{2}$? $-\frac{3}{2}$?

21. To what base is 2 the logarithm of 9? of 27? of $\frac{1}{9}$?

22. To what base is $\frac{1}{2}$ the logarithm of 5? of $\sqrt{5}$? of $\frac{1}{5}$?

§ 12.

23. If $a=1$, $b=-3$, $c=5$, find the value of:

$$\frac{a^2b^2+1}{a^2+b^2}-\frac{1-a^2c^2}{a^2-c^2}+\frac{2b^2-4ac}{b^2-c^2}-\frac{a^2+2ab+b^2}{b^2-2bc+c^2}.$$

24. If $a=25$, $b=9$, $c=-4$, $d=-1$, find the values of:

$$\sqrt{a^2}-2\sqrt[3]{b^3}+3\sqrt[4]{c^4}-4\sqrt[5]{d^5};$$

$$\sqrt{-bc}+3\sqrt{acd}-4\sqrt{-b^2d}+\sqrt{-c^2d}.$$

25. If $a=0$, $b=-2$, $c=4$, $d=-6$, find the value of:

$$3\sqrt[3]{(2b^2-a)}+2\sqrt[3]{(b^2+c^2+7)}-\sqrt[3]{[2(b+c)^2+(d+b)^2+bc]}.$$

26. If $a=2$, $x=16$, find the values of:

$\log_a x$, $\log_a\sqrt{x}$, $\log_a x^2$, $\log_a(\log_a x)$, $\log_a[\log_a(\log_a x)]$,

$\log_x a$, $\log_x\sqrt{a}$, $\log_x a^2$, $\log_x(\log_x a)$, $\log_x[-\log_x(\log_x a)]$.

27. In Ex. 23–26 show which expressions are algebraic, which transcendental, which entire, which fractional, which rational, which irrational: first in form, second in value. Show what portions of them are symmetric, and as to which letters.

§ 13.

28. If $\phi(x)\equiv x^2+3x+6$, write the expressions for:

$\phi(a)$, $\phi(2a)$, $\phi(-y)$, $\phi(x+y)$, $\phi(x-y)$,

and find the values of $\phi(0)$, $\phi(1)$, $\phi(-2)$.

29. If $\phi(x,y,z,t)=x^2+3yz+t^2$, write the expressions for:

$\phi(a^2,m,n,l)$, $\phi(0,1,-2,x)$, $\phi(x,x,x,x)$, $\phi(t,x,y,z)$.

30. As to what letters, if any, is each of the following functions symmetric?

$\phi(x+y)$; $\mathrm{F}(xy,x+y)$; $f(x,y,z)+f(y,z,x)+f(z,x,y)$;

$\phi(x)+\phi(y)+\phi'(z)$; $\mathrm{F}(x,yz)+\mathrm{F}(y,zx)+\mathrm{F}(z,xy)$.

§ 14.

31. Show what factors must be taken as coefficients in order that the following sets of terms shall be like:

$3ab$, $3bc$; $5ax^2$, $2axy$; $7mn^2$, $4m\sqrt{n}$;

$2abc$, $3bcd$, $4cdx$; $\frac{3}{4}xy^2$, $\frac{7}{8}yz\sqrt{(x^2+y^2)}$.

§ 15.

32. In Ex. 23–25 state the degree of each one of the expressions, and show which of them are homogeneous.

II. PRIMARY OPERATIONS.

§ 1. LOGICAL TERMS.

A DEFINITION is a statement of the sense in which a word or symbol is used.

A *theorem* is a general truth: if self-evident, it is an *axiom;* if auxiliary to a following theorem, it is a *lemma;* if an obvious consequence of a previous theorem, it is a *corollary.*

A theorem consists of two parts, the *hypothesis* or *data*, and the *conclusion* which, if not self-evident, is to be established by a *demonstration.*

A *converse* of a theorem is another theorem that has for data the conclusion, or the conclusion and any of the data, of the first theorem, and for conclusion some datum of the first theorem.

E.g., the theorem "If from equal numbers equals be subtracted, the remainders are equal," is an axiom, wherein the clause before the comma is the hypothesis, and the clause after the comma is the conclusion. It needs no demonstration. Its converses are: "If the remainders be equal, the numbers from which equals are subtracted are equal," and "If the remainders be equal, the numbers subtracted from equals are equal."

Of demonstrations three kinds are found in Algebra:

(*a*) *Direct proof*, wherein the conclusion follows as a direct and necessary consequence of certain axioms and definitions, and of other theorems already proved.

(*b*) *Proof by exclusion*, also called *reductio ad absurdum*, or *indirect proof*, wherein are first enumerated all possible conclusions from the given data, and then the truth of one of them is established by the exclusion as absurd of all the rest.

(*c*) *Proof by induction*, which consists of three steps:

1. Proof, either direct or indirect, that the theorem is true when applied to one or more cases at the beginning of a series of particular cases of the general theorem.

2. Proof that, if the theorem be true up to any one case inclusive, then it must also be true for the next higher case in the series.

3. Proof by progressive steps that, since, beginning with the cases actually proved (1), it is true for the next, and the next, and the next, indefinitely (2), therefore it is universally true.

A *problem* is anything to be done; usually, in Algebra, it is to find numbers or expressions that will satisfy given conditions. These numbers or expressions, together with the process of finding them, constitute the *solution* of the problem.

A solution is $\left\{\begin{matrix}\textit{general}\\ \textit{particular}\end{matrix}\right.$ when it gives $\left\{\begin{matrix}\text{all}\\ \text{some}\end{matrix}\right.$ of the numbers, or expressions, or sets of numbers or expressions, that satisfy the given conditions. Usually the general solution is sought, with a demonstration showing, by previous theorems and problems, that the solution satisfies the given conditions and is general.

A *check*, or *test*, is a comparison of results designed to detect any accidental errors in the work.

A *postulate* assumes as self-evident that the solution of a problem is possible.

The letters $\left\{\begin{matrix}\text{Q. E. D.}\\ \text{Q. E. F.}\end{matrix}\right.$ at the end of a $\left\{\begin{matrix}\text{demonstration}\\ \text{solution}\end{matrix}\right.$ stand for *quod erat* $\left\{\begin{matrix}\textit{demonstrandum}\\ \textit{faciendum}\end{matrix}\right.$ — which was to be $\left\{\begin{matrix}\text{proved.}\\ \text{done.}\end{matrix}\right.$

§ 2. COMBINATORY PROPERTIES OF OPERATIONS.

AN ALGEBRAIC OPERATION is an act by which two or more numbers, the *elements*, are combined together to produce one number, the *result*.

Manifestly, the result is a function of the elements.

An operation is $\left\{\begin{matrix}\textit{simple}\\ \textit{complex}\end{matrix}\right.$ when $\left\{\begin{matrix}\text{only}\\ \text{more than}\end{matrix}\right.$ two elements are combined. If a complex operation consist of two or more simple operations, and if they be all of the same kind, it is a *continued* operation.

E.g., the continued addition of three numbers consists of first adding two of them, and then adding the third number to this sum.

Of the two elements of a simple operation, one, the *operand*, is conceived of as acted upon by the other, the *operator*, in a way shown by the sign of operation.

E.g., in $6+2=8$, $6-2=4$, $6\times 2=12$, $6:2=3$, the operand is 6; the operator is 2; the results are 8, 4, 12, 3; the operations are addition, subtraction, multiplication, division; and the signs of operation are $+$, $-$, $\times$, $:$.

So, in $16^2=256$, $\sqrt[2]{16}$ or $\sqrt{16}=\pm 4$, $\log_2 16=4$, the operand is 16; the operator is 2; the results are 256, ± 4, 4; the operations are involution, evolution, the finding of a logarithm; and the signs of operation are, the position of the exponent, $\sqrt{\ }$, the word "log."

An operation is $\begin{cases}\textit{uni-determinate,}\\ \textit{multi-determinate,}\\ \textit{indeterminate,}\end{cases}$ when, from given elements,

it gives $\begin{cases}\text{only one result.}\\ \text{several different results, but none intermediate.}\\ \text{an infinite number of results in a continuous series.}\end{cases}$

The *rational* operations (addition, subtraction, multiplication, division, and involution to integral powers) and the finding of logarithms are generally unideterminate; but evolution is generally multideterminate; and operations with special elements are often indeterminate.

E.g., $6+2, =8$; $6-2, =4$; $6\times 2, =12$; $6:2, =3$; $3^2, =9$; $\log_3 9, =2$; are unideterminate;

but $\sqrt[2]{9}$, = either $+3$ or -3, is multideterminate;

and $0:0$, 0^0, $\log_0 0$, $\log_1 1$, are indeterminate.

When the result and the $\begin{cases}\text{operator}\\ \text{operand}\end{cases}$ are given, the $\begin{cases}\text{operand}\\ \text{operator}\end{cases}$ may be found by an operation called the $\begin{cases}\textit{first}\\ \textit{second}\end{cases}$ *inverse* of the original or *direct* operation,

wherein the operand, operator, and result

are the $\begin{cases}\text{result, operator, and operand,}\\ \text{result, operand, and operator,}\end{cases}$ respectively, of the direct operation. Hence an inverse operation is the undoing of what was done by the direct operation, and it ends where the direct operation began.

An inverse operation may be defined as an operation "the effect of which the direct operation simply annuls" It consists not in any new procedure, "but in a series of guesses suggested by prior general knowledge of the results of the direct operation, and tested by the direct operation itself." — *Boole*.

E.g., $6-2=4 \because 4+2=6$;

$6:2=3 \because 3\times 2=6$;

$\sqrt{9}=\pm 3 \because (^+3)^2=9$ and $(^-3)^2=9$;

$\log_3 9=2 \because 3^2=9$;

An inverse operation is, therefore, described by the two words "guess" and "test." The error of one guess helps the next one.

E.g., To divide 756 by 27:

Guess 30; that is too large, for the product, 27×30, is 810, which is larger than 756.

Guess 20; that is too small, for the product, 27×20, is 540, and the remainder, 216, is larger than 27.

Guess 8 as the quotient of the remainder $216:27$; this guess is right, for the product, 27×8, is 216; and the whole quotient is 28, the sum of 20 and 8.

An inverse operation may or may not be multideterminate when the direct operation is unideterminate; and the two inverses may or may not be of the same kind.

E.g., *Direct Operations.*	*First Inverses.*	*Second Inverses.*
$6+2=8$,	$8-2=6$,	$8-6=2$;
$6\times 2=12$,	$12:2=6$,	$12:6=2$;
$6^2=36$,	$\sqrt{36}=\pm 6$,	$\log_6 36=2$;

wherein the two inverses of addition are both subtraction and unideterminate;

and of multiplication, they are both division and unideterminate;

but of involution, the first is evolution and multideterminate, and the second is the finding of a logarithm and unideterminate.

A direct simple operation is sometimes the repetition of more elementary operations.

E.g., addition of a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ integer, $\pm m$, is counting a unit $\left\{\begin{matrix}\text{on}\\ \text{off}\end{matrix}\right.$ m times; and addition of a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ fraction, $\pm \frac{m}{n}$.

is counting n times $\left\{\begin{matrix}\text{on}\\ \text{off}\end{matrix}\right.$ such a number as, if counted on n times, would give a unit.

So multiplication by a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ integer, $\pm m$, is $\left\{\begin{matrix}\text{adding}\\ \text{subtracting}\end{matrix}\right.$ the multiplicand m times $\left\{\begin{matrix}\text{to}\\ \text{from}\end{matrix}\right.$ 0; and multiplication by a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ fraction, $\pm \frac{m}{n}$, is $\left\{\begin{matrix}\text{adding}\\ \text{subtracting}\end{matrix}\right.$ m times $\left\{\begin{matrix}\text{to}\\ \text{from}\end{matrix}\right.$ 0 such a number as, if added n times to 0, would give the multiplicand.

So, involution by a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ integral exponent, $\pm m$, is $\left\{\begin{matrix}\text{multiplying}\\ \text{dividing}\end{matrix}\right.$ 1 by the base m times; and involution by a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ fractional exponent, $\pm \frac{m}{n}$, is $\left\{\begin{matrix}\text{multiplying}\\ \text{dividing}\end{matrix}\right.$ 1, m times, by such a number as, if multiplied n times into 1, would give the base.

Thus the operations of addition, multiplication, and involution all come from the more elementary operation of counting.

So, often, when the operator is a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ integer, as $\pm m$, then the more elementary $\left\{\begin{matrix}\text{direct}\\ \text{inverse}\end{matrix}\right.$ operation is performed m times in succession upon the operand; and when the operator is a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ fraction, as $\pm \frac{m}{n}$, then some operation is performed m times which, if performed n times, would be equivalent to the more elementary $\left\{\begin{matrix}\text{direct}\\ \text{inverse}\end{matrix}\right.$ operation.

The *modulus* of a simple operation is that operator, if any, which always makes the result equal to the operand.

E.g., $\because x + 0 = x$ and $x - 0 = x$, [x any number

$x \times 1 = x$ and $x : 1 = x$,

and $x^1 = x$ and $\sqrt[1]{x} = x$;

$\therefore$ the modulus of addition and subtraction is 0,

the modulus of multiplication and division is 1,

and the modulus of involution and evolution is 1.

An operation is $\left\{ \begin{array}{l} \textit{commutative} \\ \textit{non-commutative} \end{array} \right.$ when the elements $\left\{ \begin{array}{l} \text{can} \\ \text{cannot} \end{array} \right.$ exchange places without changing the result; *i.e.*, when the result is $\left\{ \begin{array}{l} \text{a symmetric} \\ \text{an unsymmetric} \end{array} \right.$ function of the elements.

E.g., *Commutative Operations.* *Non-commutative Operations.*

$$2+3=3+2, \qquad 2-3\neq 3-2,$$
$$2\times 3=3\times 2. \qquad 2:3\neq 3:2,$$
$$2^3 \neq 3^2,$$
$$\sqrt[3]{2} \neq \sqrt[2]{3},$$
$$\log_3 2\neq \log_2 3.$$

A continued operation is $\left\{ \begin{array}{l} \textit{associative} \\ \textit{non-associative} \end{array} \right.$ when, as long as the elements do not exchange places, they $\left\{ \begin{array}{l} \text{can} \\ \text{cannot} \end{array} \right.$ be grouped at will without changing the result.

E.g., *Associative Operations.* *Non-associative Operations.*

$$(12+4)+2=12+(4+2), \qquad (12-4)-2\neq 12-(4-2),$$
$$(12\times 4)\times 2=12\times(4\times 2), \qquad (12:4):2\neq 12:(4:2),$$
$$(12^4)^2 \neq 12^{(4^2)}.$$

A second operation is $\left\{ \begin{array}{l} \textit{distributive} \\ \textit{non-distributive} \end{array} \right.$ *as to* a first operation when the final result $\left\{ \begin{array}{l} \text{is} \\ \text{is not} \end{array} \right.$ the same, whether the second operator act upon the result of the first operation, or upon the separate elements of the first operation, and then these results are combined by the first operation. An operation distributive as to addition is also called *linear*.

E.g., *Distributive Operations.* *Non-distributive Operations.*

Multiplication as to addition.

$$\overline{12+6}\times 3=\overline{12\times 3}+\overline{6\times 3};$$

Addition as to multiplication.

$$\overline{12\times 6}+3\neq\overline{12+3}\times\overline{6+3}.$$

Involution as to multiplication.

$$\overline{12\times 6}^2 = 12^2\times 6^2;$$

Involution as to addition.

$$\overline{12+6}^2 \neq 12^2+6^2;$$

Evolution as to multiplication.

$$\sqrt[3]{27\times 8} = \sqrt[3]{27}\times\sqrt[3]{8};$$

Evolution as to addition.

$$\sqrt[3]{27\pm 8} \neq \sqrt[3]{27}\pm\sqrt[3]{8};$$

Finding of logarithms as to addition.

$$\log_6(216+36)\neq\log_6 216+\log_6 36;$$

Finding of logarithms as to multiplication.

$$\log_6(216\times 36)\neq\log_6 216\times\log_6 36.$$

§ 3. AXIOMS.

1. Numbers equal to the same number are equal to each other.

2. If to equal numbers equals be added, the SUMS are equal.

3. If from equal numbers equals be subtracted, the REMAINDERS are equal.

4. If equal numbers be multiplied by equals, the PRODUCTS are equal.

5. If equal numbers be divided by equals, the QUOTIENTS are equal.

6. If equal numbers be raised to like integral powers, the POWERS are equal.

7. If of two equal numbers like roots be taken, every ROOT of the first number is equal to some root of the other.

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8. If of three numbers the first be $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than the second, and the second be equal to or $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than the third, then is the first $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than the third.

9. If one number be $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than another, and if to each of them be added the same number or equal numbers, then is the first sum $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than the other.

10. If one number be $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than another, and if from each of them be subtracted the same number or equal numbers, then is the first remainder $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than the other.

11. If one number be $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than another, and if each of them be subtracted from the same number or from equal numbers, then is the first remainder $\left\{\begin{array}{l}\text{less}\\\text{greater}\end{array}\right.$ than the other.

12. If one set of numbers be $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than another set of as many more, each than each, then is the sum of the first set $\left\{\begin{array}{l}\text{greater}\\\text{less}\end{array}\right.$ than the sum of the others.

13. If one number be $\left\{ \begin{matrix} \text{greater} \\ \text{less} \end{matrix} \right.$ than another, and if each of them be multiplied or divided by the same or equal positive numbers, then is the first product or quotient $\left\{ \begin{matrix} \text{greater} \\ \text{less} \end{matrix} \right.$ than the other.

14. If one number be $\left\{ \begin{matrix} \text{greater} \\ \text{less} \end{matrix} \right.$ than another, and if each of them be multiplied or divided by the same or equal negative numbers, then is the first product or quotient $\left\{ \begin{matrix} \text{less} \\ \text{greater} \end{matrix} \right.$ than the other.

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15. If of three numbers the first be $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than the second, and the second be equal to or $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than the third, then is the first $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than the third.

16. If one number be $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than another, and if each of them be multiplied by the same number or by equal numbers, then is the first product $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than the other.

17. If one number be $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than another, and if each of them be divided by the same number or by equal numbers, then is the first quotient $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than the other.

18. If one number be $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than another, and if the same number or equal numbers be divided by each of them, then is the first quotient $\left\{ \begin{matrix} \text{smaller} \\ \text{larger} \end{matrix} \right.$ than the other.

19. If one set of numbers be $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than another set of as many more, each than each, then is the product of the first set $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than the product of the others.

20. If one number be $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than another, and if like positive powers or roots of them be taken, then is the first power or root $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ than the other.

21. If one number be $\left\{\begin{matrix}\text{larger}\\ \text{smaller}\end{matrix}\right.$ than another, and if like negative powers or roots of them be taken, then is the first power or root $\left\{\begin{matrix}\text{smaller}\\ \text{larger}\end{matrix}\right.$ than the other.

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22. If two numbers be opposites, one of them is positive and the other is negative; they are equally large; and their sum is naught.

23. If all the letters of an entire expression stand for integers, the value of the expression is an integer.

NOTE 1. For convenience, and because quite evident, all the propositions above given are called axioms, although, in strictness, some of them are deducible from others.

E.g., Ax. 1 is deducible from Ax. 8.

For let A, B, C be three numbers such that $A = C$ and $B = C$;
then either $A = B$, or $A > B$, or $A < B$; and of these three, the only possible conclusions from the data, one must be true, and the others false.

Suppose $A > B$;
then $\because A > B$ and $B = C$, [hyp.
$\therefore A > C$, [ax. 8
a consequence from the supposition $A > B$, which is contrary to the hypothesis of the theorem, and therefore absurd.

$\therefore$ the supposition $A > B$, which led to this absurd consequence, is itself absurd; and $A \ngtr B$.

So it may be proved that $A \nless B$;
and $\because A \ngtr B$ and $A \nless B$,
$\therefore$ it is only left that $A = B$. Q. E. D.

NOTE 2. The reader will observe that the form of statement is different in Ax. 7 from that in any of the others. It will appear later that in general a number has two square roots not equal to each other, three cube roots not equal, and so on.

☞ The theorems that follow in this chapter, though universally true, are here proved for commensurable numbers only: for incommensurables see VII. § 7, and for imaginaries, see X. §§ 2, 3.

§ 4. ADDITION COMMUTATIVE AND ASSOCIATIVE.

THEOR. 1. *The sum of two or more numbers is the same, in whatever order the numbers are added, and however they are grouped.*

Let ${}^+a$, ${}^+b$, ... be any positive integers, ${}^-m$, ${}^-n$, ... any negative integers, ${}^+\left(\frac{r}{x}\right)$, ${}^-\left(\frac{s}{y}\right)$, ... any simple fractions; then will:

$$a + b + \cdots + {}^-m + {}^-n + \cdots + {}^+\left(\frac{r}{x}\right) + {}^-\left(\frac{s}{y}\right) \cdots$$

$$= {}^+\left(\frac{r}{x}\right) + {}^-\left(\frac{s}{y}\right) + \overline{a + {}^-m} + \overline{b + {}^-n} + \cdots$$

$= \cdots$ whatever the order or grouping of the terms.

For $\because$ ${}^+a = \ 1 + 1 + 1 + \cdots$ a times counted on (forward),

${}^+b = \ 1 + 1 + 1 + \cdots$ b times counted on, and so on,

and $\because$ ${}^-m = -1 - 1 - 1 - \cdots$ m times counted off (backward),

${}^-n = -1 - 1 - 1 - \cdots$ n times counted off, and so on,

and $\because$ ${}^+\left(\frac{r}{x}\right) = +\frac{1}{x} + \frac{1}{x} + \frac{1}{x} + \cdots$ r times counted on,

${}^-\left(\frac{s}{y}\right) = -\frac{1}{y} - \frac{1}{y} - \frac{1}{y} - \cdots$ s times counted off, and so on,

$\therefore$ the whole collection of units and parts of units, being the same, counts the same, on and off, whichever unit or group of units, part or group of parts, is counted first, whichever second, and so on;

and that, whether the units, or parts of units, be of the same value, or of different values. Q. E. D.

SUM OF OPPOSITES.

COR. 1. *The sum of the opposites of two or more numbers is the opposite of their sum.*

For let ${}^+\mathrm{A}$, ${}^+\mathrm{B}$, ... be any positive numbers, integral or fractional; and ${}^-\mathrm{M}$, ... any negative numbers;

then $\because$ $\overline{{}^-\mathrm{A} + {}^-\mathrm{B} + \cdots + {}^+\mathrm{M} + \cdots} + \overline{{}^+\mathrm{A} + {}^+\mathrm{B} + \cdots + {}^-\mathrm{M} + \cdots}$

$= \overline{{}^-\mathrm{A} + {}^+\mathrm{A}} + \overline{{}^-\mathrm{B} + {}^+\mathrm{B}} + \cdots + \overline{{}^+\mathrm{M} + {}^-\mathrm{M}} + \cdots = 0,$ [th.

$\therefore$ $\overline{{}^-\mathrm{A} + {}^-\mathrm{B} + \cdots + {}^+\mathrm{M} + \cdots}$ and $\overline{{}^+\mathrm{A} + {}^+\mathrm{B} + \cdots + {}^-\mathrm{M} + \cdots}$ are opposites. Q. E. D. [I. § 6 df.

THEORY OF SUBTRACTION.

COR. 2. *If to the minuend the opposite of the subtrahend be added, the sum is the remainder.*

For let M, S, and R $\equiv$ minuend, subtrahend, and remainder;

then $\because$ M $=$ S $+$ R, [I. § 7 df.

$\therefore$ M $+(-$S$)=$ R $+$ S $+(-$S$)$ [ax. 2

$=$ R $+($S $-$ S$)$ [th.

$=$ R $+ 0 \quad =$ R. Q. E. D.

COR. 3. *If there be a series of additions and subtractions, the final result is the same, in whatever order they are performed, and however the elements are grouped; but, whenever any group is made to follow the sign of subtraction, the sign of each element of the group is reversed.*

§ 5. SIGN OF PRODUCT.

THEOR. 2. *If the multiplier be positive, the product is of the same sense as the multiplicand; if negative, of the opposite sense.*

For $\because$ multiplication by a $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right.$ multiplier is a repeated $\left\{\begin{matrix}\text{addition}\\\text{subtraction}\end{matrix}\right.$ of the multiplicand, or of one of the equal parts of it, $\left\{\begin{matrix}\text{to}\\\text{from}\end{matrix}\right.$ the modulus 0, [I. § 8 df.

and $\because$ subtraction of any number from 0 gives the opposite of that number; [th. 1 cr. 3

$\therefore$ the sense of the $\left\{\begin{matrix}\text{sum}\\\text{remainder}\end{matrix}\right.$ (product) $\left\{\begin{matrix}\text{is not}\\\text{is}\end{matrix}\right.$ changed thereby. Q. E. D.

COR. 1. *If the number of negative factors be* $\left\{\begin{matrix}\textit{even,}\\\textit{odd,}\end{matrix}\right.$ *the product is* $\left\{\begin{matrix}\textit{positive.}\\\textit{negative.}\end{matrix}\right.$

NOTE. In this corollary 0 is counted an even number.

COR. 2. *In division, if the divisor be* $\left\{\begin{matrix}\textit{positive,}\\\textit{negative,}\end{matrix}\right.$ *the quotient is of the* $\left\{\begin{matrix}\textit{same}\\\textit{opposite}\end{matrix}\right.$ *sense* $\left\{\begin{matrix}\textit{as}\\\textit{to}\end{matrix}\right.$ *the dividend.*

NOTE. Th. 2 and Cor. 2 are summarized in the familiar rule for sign of product or quotient: "Like signs give $+$; unlike, $-$."

§ 6. MULTIPLICATION COMMUTATIVE AND ASSOCIATIVE.

THEOR. 3. *The product of two or more numbers is the same, in whatever order the factors are multiplied, and however they are grouped.*

(*a*) *Two positive integers*, a, b; *then will* $a \times b = b \times a$.

For let

$$\begin{array}{ccccc} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ * & * & * & \cdots & * \end{array}$$

be a collection of stars, trees, or any other units, consisting of a horizontal lines, and b vertical columns;

then ∵ if a, the number of stars in one column, be multiplied by b, the number of columns, the product, $a \times b$, gives the whole number in the collection;

and ∵ if b, the number of stars in one line, be multiplied by a, the number of lines, the product, $b \times a$, gives the whole number in the collection;

∴ $a \times b = b \times a$. Q. E. D. [ax. 1

(*b*) *Three positive integers*, a, b, c; *then will*

$$\overline{a \times b} \times c = \overline{b \times a} \times c = c \times \overline{a \times b} = c \times \overline{b \times a}$$
$$= \overline{a \times c} \times b = \overline{c \times a} \times b = b \times \overline{a \times c} = b \times \overline{c \times a}$$
$$= \overline{b \times c} \times a = \overline{c \times b} \times a = a \times \overline{b \times c} = a \times \overline{c \times b}.$$

For let

$$\begin{array}{ccccc} a & a & a & \cdots & a \\ a & a & a & \cdots & a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a & a & a & \cdots & a \end{array}$$

be a collection of groups of a units each, in b horizontal lines and c vertical columns;

and ∵ $a \times b$ is the number of units in one column,

∴ $\overline{a \times b} \times c$ is the number of units in all the c columns, *i.e.*, in the whole collection;

then ∵ $a \times c$ is the number of units in one line,

∴ $\overline{a \times c} \times b$ is the number of units in all the b lines, *i.e.*, in the whole collection;

∴ $\overline{a \times b} \times c = \overline{a \times c} \times b$. Q. E. D. [ax. 1

So let

$$\begin{array}{ccccc} b & b & b & \dots & b \\ b & b & b & \dots & b \\ \dots & \dots & \dots & \dots & \dots \\ b & b & b & \dots & b \end{array}$$

be a collection of groups of b units each, in a lines and c columns;

then $\because$ each of the b columns has $b \times a$ units,

and $\because$ each of the a lines contains $b \times c$ units,

$\therefore \overline{b \times a} \times c = \overline{b \times c} \times a$. Q. E. D.

But $\because$ $a \times b = b \times a$, $a \times c = c \times a$, $b \times c = c \times b$,

and $\overline{a \times b} \times c = c \times \overline{a \times b}$, $\overline{a \times c} \times b = b \times \overline{a \times c}$,

$\overline{b \times c} \times a = a \times \overline{b \times c}$; [(a)

$\therefore \overline{a \times b} \times c = \overline{b \times a} \times c = c \times \overline{a \times b} = c \times \overline{b \times a}$

$= \overline{a \times c} \times b = \overline{c \times a} \times b = b \times \overline{a \times c} = b \times \overline{c \times a}$

$= \overline{b \times c} \times a = \overline{c \times b} \times a = a \times \overline{b \times c} = a \times \overline{c \times b}$. Q. E. D.

NOTE. In this proof the reader will observe that the bar can be placed over a group of factors, or removed, at pleasure, when that group stands at the left end of the series, but not otherwise.

E.g., $a \times b \times c \equiv \overline{a \times b} \times c$;

for in either case, the product of a by b is first found and then that product is multiplied by c.

But $a \times \overline{b \times c}$ is a very different matter;

for in this case, the product of b by c is first found, and a is then multiplied by this product.

(c) *Any number of positive integers.*

1. The theorem is true for two factors, and for three. [(a, b)

2. If it be true up to n factors inclusive, it is true also for $n+1$ factors.

For let the $n+1$ factors, $a, b, c, \dots i, j, k$, be grouped and multiplied together in any desired way, and let the product be P;

then $\because$ P is got by multiplying the product, say Q, of some of these factors, by the remaining factor, or by the product, say R, of the remaining factors,

$\therefore$ P = Q $\times$ R.

Let R be that one of these products which has the factor K,
and let S be the product of the other factors of R;
then ∵ neither Q, R, S nor the product Q × S has more than n factors,

∴ in each of them the several factors may take any desired order, [hyp. above

$$\therefore \text{P}, = \text{Q} \times \text{R}, = \text{Q} \times \overline{\text{S} \times k}$$
$$= \overline{\text{Q} \times \text{S}} \times k \qquad [(b)$$
$$= \overline{a \times b \times \cdots \times j} \times k$$
$$= a \times b \times \cdots \times j \times k.$$

3. But the theorem is true for three factors, [(a, b)

∴ it is true for four factors; [2, above

So for five factors, for six factors, ... Q. E. D.

NOTE. This proof is an example of proof by induction. [§ 1

(*d*) *Any number of integers*, $^+$a, $^-$b, $^+$c, ..., $^+$l, $^-$m, $^-$n, *whereof* k *factors are negative.*

For in whatever order the factors are multiplied,

P, their product, $= \pm\, {}^+a \times {}^+b \times \cdots \times {}^+m \times {}^+n$, [$(c)$

a $\left\{ \begin{array}{l} \text{positive} \\ \text{negative} \end{array} \right.$ number when k is $\left\{ \begin{array}{l} \text{even.} \\ \text{odd.} \end{array} \right.$ Q. E. D. [th. 2 cr. 1

(*e*) *Any number of factors*, a, b, c, ... $\frac{\text{r}}{\text{x}}, \frac{\text{s}}{\text{y}}, \ldots$, *whereof some, or all, are simple fractions.*

For ∵ in the product $a \times b \times c \times \cdots \times \frac{r}{x} \times \frac{s}{y} \times \cdots$,

$$= \pm\, {}^+a \times {}^+b \times {}^+c \times \cdots \times {}^+\!\left(\frac{r}{x}\right) \times {}^+\!\left(\frac{s}{y}\right) \times \cdots,$$

each unit of the product $a \times b \times c \times \cdots$ is divided into x equal parts and r of them are taken,

and then each of these $a \times b \times c \times \cdots \times r$ parts is divided into y equal parts, and s of them are taken, and so on,

i.e., the unit is divided into $x \times y \times \cdots$ equal parts, and $a \times b \times c \times \cdots \times r \times s \times \cdots$ of them are taken;

$$\therefore a \times b \times c \times \cdots \times \frac{r}{x} \times \frac{s}{y} \times \cdots = \frac{a \times b \times c \times \cdots \times r \times s \cdots}{x \times y \times \cdots}.$$

So in the product $a \times \frac{s}{y} \times b \times \frac{r}{x} \times c \times \cdots$ each of the a units is divided into y parts, and s of them are taken,

and then each of these $a \times s$ parts is taken b times,
and then each of these $a \times s \times b$ parts is divided into x equal parts, and r of them are taken,
and then each of these $a \times s \times b \times r$ parts is taken c times, and so on;
i.e., the unit is divided into $y \times x \times \cdots$ equal parts,
and $a \times s \times b \times r \times c \times \cdots$ of them are taken.

$$\therefore\ a \times \frac{s}{y} \times b \times \frac{r}{x} \times c \times \cdots = \frac{a \times s \times b \times r \times c \times \cdots}{y \times x \times \cdots}.$$

But $\because$ $y \times x \times \cdots = x \times y \times \cdots$, [$(d)$
and $a \times s \times b \times r \times c \times \cdots = a \times b \times c \times \cdots \times r \times s$;
$\therefore$ in each of these two products, the unit is divided into the same number of equal parts, and the same number of these parts are taken.
So for all other possible products of these factors;
$\therefore$ the products are all equal. Q. E. D.

(f) *Any number of factors, whereof some or all are neither integers nor simple fractions, but which are all commensurable.*

For, let A, B, C, ... $\frac{\text{R}}{\text{X}}$, $\frac{\text{S}}{\text{Y}}$, ... be the values of such factors when reduced, wherein the letters all stand for integers; [I. § 1 df.
then $\because$ the value of the product $\text{A} \times \text{B} \times \text{C} \times \cdots \times \frac{\text{R}}{\text{X}} \times \frac{\text{S}}{\text{Y}} \times \cdots$ is the same, whatever the order or grouping of the factors, [(d, e)
$\therefore$ the product of the given factors is the same, etc. Q. E. D.

THEORY OF FRACTIONS.

COR. 1. *The product of two or more fractions is a fraction whose* $\left\{\begin{matrix}\textit{numerator}\\ \textit{denominator}\end{matrix}\right.$ *is the product of the* $\left\{\begin{matrix}\textit{numerators}\\ \textit{denominators}\end{matrix}\right.$ *of the given fractions.*

Let $\frac{n}{d}, \frac{n'}{d'}, \frac{n''}{d''}, \ldots$ be any fractions,

then will $\frac{n}{d} \times \frac{n'}{d'} \times \cdots = \frac{n \times n' \times \cdots}{d \times d' \times \cdots}$.

(a) n, d, ... *all integers.*
This case was proved in the demonstration of Th. 3 (e).

(*b*) *The fractions and their numerators and denominators any commensurables whatever.*

For, let $f, f', \ldots$ be the values of $\frac{n}{d}, \frac{n'}{d'}, \ldots$ respectively;

then $\because\ n = f \times d,\quad n' = f' \times d', \ldots,$ [I. § 9 df.

$\therefore\ n \times n' \times \cdots = (f \times d) \times (f' \times d') \times \cdots$ [ax. 4

$= (f \times f' \times \cdots) \times (d \times d' \times \cdots),$ [th. 3

$$\therefore\ \frac{n \times n' \times \cdots}{d \times d' \times \cdots} = f \times f' \times \cdots$$

$$= \frac{n}{d} \times \frac{n'}{d'} \times \cdots.$$ Q.E.D. [I. § 9 df.

COR. 2. *The reciprocal of any fraction is the same fraction inverted;* i.e., *a fraction whose* $\left\{ \begin{array}{l} \textit{numerator} \\ \textit{denominator} \end{array} \right.$ *is the* $\left\{ \begin{array}{l} \textit{denominator} \\ \textit{numerator} \end{array} \right.$ *of the given fraction.*

For, let n and d be any commensurable numbers;

then $\because\ \frac{d}{n} \times \frac{n}{d} = \frac{d \times n}{n \times d} = 1,$

$\therefore\ \frac{d}{n}$ is the reciprocal of $\frac{n}{d}$. Q.E.D. [I. § 8 df.

COR. 3. *The quotient of one number by another is the product of the first by the reciprocal of the other.*

(*a*) *The divisor and dividend both simple fractions.*

For, let $\frac{n}{d}$ and $\frac{n'}{d'}$ be any two fractions,

then $\because\ \left(\frac{n}{d} \times \frac{d'}{n'}\right) \times \frac{n'}{d'} = \frac{n}{d} \times \left(\frac{d'}{n'} \times \frac{n'}{d'}\right) = \frac{n}{d} \times \frac{d' \times n'}{n' \times d'} = \frac{n}{d};$

$\therefore\ \frac{n}{d} : \frac{n'}{d'} = \frac{n}{d} \times \frac{d'}{n'}.$ Q.E.D. [I. § 9 df.

NOTE. Cors. 1, 2, 3 lead to the reduction of complex products and quotients of commensurables to simple fractions, either directly or by progressive steps. By Cor. 2, the reciprocal of any commensurable is a commensurable; hence, by Cor. 3, if both numerator and denominator be commensurables, so is the fraction; by Cors. 1, 3, any product or quotient of such fractions is got as a simple fraction, and so on. Compare Th. 5, Cor. 8.

(*b*) *The dividend and divisor any commensurables whatever.*

For, let N, D be any two commensurables;
then $\because$ D is commensurable,

$\therefore \frac{1}{\text{D}}$ is likewise commensurable; [(*a*) nt.

$\therefore \left(\text{N} \times \frac{1}{\text{D}}\right) \times \text{D} = \text{N} \times \left(\frac{1}{\text{D}} \times \text{D}\right) = \text{N}$; [th. 3

$\therefore \text{N} \times \frac{1}{\text{D}} = \text{N} : \text{D}$. Q. E. D. [I. § 9 df.

COR. 4. *The product of the reciprocals of two or more numbers is the reciprocal of their product.*

For, let $a, b, \ldots \frac{r}{x}, \frac{s}{y}, \ldots$ be any numbers,

$$\text{then} \because \left(\frac{1}{a} \times \frac{1}{b} \times \cdots \times \frac{x}{r} \times \frac{y}{s} \times \cdots\right) \times \left(a \times b \times \cdots \times \frac{r}{x} \times \frac{s}{y} \times \cdots\right)$$
$$= \left(\frac{1}{a} \times a\right) \times \left(\frac{1}{b} \times b\right) \times \cdots \times \left(\frac{x}{r} \times \frac{r}{x}\right) \times \left(\frac{y}{s} \times \frac{s}{y}\right) \times \cdots$$
$$= 1;$$

$\therefore \frac{1}{a} \times \frac{1}{b} \times \cdots \times \frac{x}{r} \times \frac{y}{s} \times \cdots$ and $a \times b \times \cdots \times \frac{r}{x} \times \frac{s}{y} \times \cdots$ are reciprocals. Q. E. D.

COR. 5. *If the numerator of a fraction be multiplied by any number, the fraction is multiplied by that number.*

For, let $\frac{n}{d}$ be any fraction, and k any multiplier;

then $$\frac{n \times k}{d} = (n \times k) \times \frac{1}{d} = \left(n \times \frac{1}{d}\right) \times k = \frac{n}{d} \times k. \quad \text{Q. E. D.}$$

NOTE. In this corollary and the two corollaries that follow, "multiplied by" includes "divided by," since to divide by k is but to multiply by its reciprocal.

COR. 6. *If the denominator of a fraction be multiplied by any number, the fraction is divided by that number.*

For, let $\frac{n}{d}$ be any fraction, and k any multiplier;

then $$\frac{n}{d \times k} = n \times \frac{1}{d \times k} = n \times \left(\frac{1}{d} \times \frac{1}{k}\right) = \left(n \times \frac{1}{d}\right) \times \frac{1}{k} = \frac{n}{d} : k.$$

COR. 7. *If both terms of a fraction be multiplied by the same number, the value of the fraction is not changed.*

For, let $\frac{n}{d}$ be any fraction, and k any multiplier;

then $$\frac{n \times k}{d \times k} = \frac{n}{d} \times \frac{k}{k} = \frac{n}{d}.$$ Q. E. D. [cr. 1

COR. 8. *If there be a series of multiplications and divisions, the final result is the same, in whatever order they are performed, and however the elements are grouped; but whenever any group is made to follow the sign of division, the sign of operation of each element of the group is reversed.*

NOTE. The reader will observe the analogy between this corollary and Th. 1, Cor. 3. He will see that, if three or more numbers are joined by the signs $\times$ and $:$, he may introduce or remove brackets just as if they were joined by the signs $+$ and $-$.

QUOTIENT OF A PRODUCT BY ITS FACTORS.

COR. 9. *If the product of several factors be divided by one of them, or by the product of two or more of them, the quotient is the product of the remaining factors.*

For $\because$ the product of the remaining factors by the divisor is the product of all the factors, [th. 3

$\therefore$ the product of the remaining factors is the quotient of the product of all the factors by the divisor. Q. E. D.

PRODUCT OF INTEGRAL POWERS.

COR. 10. *The product of two or more integral powers of any same number is a power of that number whose exponent is the sum of the exponents in the factors.*

For, let A be any number, and $l, m, n \ldots$ any positive integers,

then $\because$ $A^l = 1 \times A \times A \times A \times \cdots l$ times,

$A^m = 1 \times A \times A \times A \times \cdots m$ times,

and $A^{-n} = 1 : A : A : A : \cdots n$ times, and so on. [I. § 10 df.

$$\begin{aligned}\therefore\ & A^l \times A^m \times A^{-n} \times \cdots \\ & = (1 \times A \times A \times \cdots l \text{ times}) \times (1 \times A \times A \times \cdots m \text{ times}) \\ & \qquad \times (1 : A : A : \cdots n \text{ times}) \times \cdots \\ & = 1 \times A \times A \times \cdots (l + m + \cdots) \text{ times} \\ & \qquad : A : A : \cdots (n + \cdots) \text{ times} \\ & = 1 \times A \times A \times \cdots (l + m + \cdots - n - \cdots) \text{ times} \quad [\text{cr. } 9 \\ & = A^{l+m-n\cdots}.\end{aligned}$$ Q. E. D.

COR. 11. *An integral power of any integral power of a base, is that power of the base whose exponent is the product of the two given exponents.*

Let A be any number, m and n any integers, then will $(A^n)^m = A^{mn}$.

(*a*) m *positive.*

For $(A^n)^{+m}, \equiv 1 \times A^n \times A^n \times \cdots m \text{ times},$ [I. § 10 df.

$= A^{n+n+\cdots m \text{ times}}$ [cr. 10

$= A^{mn}$. Q. E. D.

(*b*) m *negative.*

For $(A^n)^{-m}, \equiv 1 : A^n : A^n : \cdots {}^+m \text{ times},$ [I. § 10 df.

$= 1 : A^{n+n+\cdots {}^+m \text{ times}}$ [cr. 10

$= 1 : A^{+mn}$

$= A^{-mn}$, or $= A^{mn}$ if the sign of quality be erased.

Q. E. D.

COR. 12. *The* $\left\{ \begin{matrix} \textit{product} \\ \textit{quotient} \end{matrix} \right.$ *of like integral powers of two or more numbers is the same power of the* $\left\{ \begin{matrix} \textit{product} \\ \textit{quotient} \end{matrix} \right.$ *of those numbers.*

Let A, B, C, ... be any numbers, n any integer, positive or negative, then will $A^n \times B^n : C^n \cdots = \overline{A \times B : C \cdots}^n$.

(*a*) n *positive.*

For $\because A^n = 1 \times A \times A \times \cdots n$ times,

$B^n = 1 \times B \times B \times \cdots n$ times,

$C^n = 1 \times C \times C \times \cdots n$ times, and so on;

$\therefore A^n \times B^n : C^n \cdots$

$= (1 \times A \times A \times \cdots n \text{ times}) \times (1 \times B \times B \times \cdots n \text{ times})$

$: (1 \times C \times C \times \cdots n \text{ times}) \cdots$

$= 1 \times \overline{A \times B : C \cdots} \times \overline{A \times B : C \cdots} \times \cdots n$ times

$= \overline{A \times B : C \cdots}^n$. Q. E. D.

(*b*) n *negative.*

For $\because A^{-n} = 1 : A : A : \cdots {}^+n$ times,

$B^{-n} = 1 : B : B : \cdots {}^+n$ times,

$C^{-n} = 1 : C : C : \cdots {}^+n$ times, and so on;

$\therefore A^{-n} \times B^{-n} : C^{-n} \cdots$

$= (1 : A : A : \cdots {}^+n \text{ times}) \times (1 : B : B : \cdots {}^+n \text{ times})$

$: (1 : C : C : \cdots {}^+n \text{ times}) \cdots$

$= 1 : \overline{A \times B : C \cdots} : \overline{A \times B : C \cdots} : \cdots {}^+n$ times [cr. 8

$= (A \times B : C \cdots)^{-n}$. Q. E. D.

§ 7. MULTIPLICATION DISTRIBUTIVE AS TO ADDITION.

THEOR. 4. *The sum of two or more like numbers is the product of the common factor by the sum of the coefficients.*

Let $^+m \cdot a$, $^+n \cdot a$, $^-p \cdot a$, $\frac{^\pm r}{x} \cdot a$, ... be any like numbers, whereof a is the common factor, and ^+m, ^+n, ^-p, $\frac{^\pm r}{x}$, ... are the coefficients;

then will $^+m \cdot a + {^+n} \cdot a + {^-p} \cdot a + \frac{^\pm r}{x} \cdot a + \cdots$

$$= ({^+m} + {^+n} + {^-p} + \frac{^\pm r}{x} + \cdots) \cdot a.$$

For $\because$

$$\begin{aligned} {^+m} \cdot a &= a + a + a + \cdots \text{ counted on } m \text{ times,} \\ {^+n} \cdot a &= a + a + a + \cdots \text{ counted on } n \text{ times,} \\ {^-p} \cdot a &= -a - a - a - \cdots \text{ counted off } p \text{ times,} \\ \frac{^\pm r}{x} \cdot a &= \text{the } x\text{th part of } a \text{ counted, on or off, } r \text{ times,} \\ \cdots \quad & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots, \end{aligned}$$

$$\begin{aligned} \therefore\ {^+m} \cdot a + {^+n} \cdot a + {^-p} \cdot a + \frac{^\pm r}{x} \cdot a + \cdots & \\ = a + a + a + \cdots \text{ counted } ({^+m} + {^+n} + {^-p} \cdots) \text{ times} & \\ \pm \text{ the } \frac{r}{x}\text{th part of } a + \cdots; & \end{aligned}$$

$\therefore$ the whole sum is $({^+m} + {^+n} + {^-p} + \frac{^\pm r}{x} + \cdots) \cdot a$. Q. E. D.

COR. *The sum of two or more fractions having a common denominator is a fraction whose numerator is the sum of their numerators, and whose denominator is the common denominator.*

For, let $\frac{n}{d}$, $\pm \frac{n'}{d}$, ... be any fractions having a common denominator,

then
$$\begin{aligned} \frac{n}{d} \pm \frac{n'}{d} + \cdots &= \left(n \times \frac{1}{d}\right) \pm \left(n' \times \frac{1}{d}\right) + \cdots && \text{[th. 3 cr. 3} \\ &= (n \pm n' + \cdots) \times \frac{1}{d} && \text{[th.} \\ &= \frac{n \pm n' + \cdots}{d}. && \text{Q. E. D.} \end{aligned}$$

NOTE. In this corollary, and in general, subtraction is but a case of addition.

E.g., to subtract the fraction $\frac{n'}{d}$ is to add its opposite, $-\frac{n'}{d}$.

THEOR. 5. *The product of two or more polynomials is the sum of the several products of each term of the first factor by each term of the second factor by each term of the third factor, and so on.*

(*a*) *Two factors,* $\mathrm{a} + \mathrm{b} + \cdots + \frac{\mathrm{r}}{\mathrm{x}} + \cdots$ *and* $\mathrm{a}' + \mathrm{b}' + \cdots + \frac{\mathrm{r}'}{\mathrm{x}'} + \cdots$, *wherein* a, b, ..., a′, b′, ... *are any integers, positive or negative, and* $\frac{\mathrm{r}}{\mathrm{x}}, \frac{\mathrm{r}'}{\mathrm{x}'}, \cdots$ *are any simple fractions.*

For $(a+b+\cdots+\frac{r}{x}+\cdots) \times (a'+b'+\cdots+\frac{r'}{x'}+\cdots)$

$= (a+b+\cdots+\frac{r}{x}+\cdots)$ counted $^+a'$ times, on or off,

$+ (a+b+\cdots+\frac{r}{x}+\cdots)$ counted $^+b'$ times, on or off,

$+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$+$ the $\frac{^+r'}{x'}$th part of $(a+b+\cdots+\frac{r}{x}+\cdots)$, on or off,

$+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$=$ a counted a' times $+$ b counted a' times $+ \cdots + \frac{r}{x}$ counted a' times $+ \cdots$

$+$ a counted b' times $+$ b counted b' times $+ \cdots + \frac{r}{x}$ counted b' times $+ \cdots$

$+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$+$ the $\frac{r'}{x'}$th part of a $+$ the $\frac{r'}{x'}$th part of $b + \cdots$

$+$ the $\frac{r'}{x'}$th part of $\frac{r}{x} + \cdots$

$+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$

$$= a \times a' + b \times a' + \cdots + \frac{r}{x} \times a' + \cdots$$

$$+ a \times b' + b \times b' + \cdots + \frac{r}{x} \times b' + \cdots$$

$$+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$+ a \times \frac{r'}{x'} + b \times \frac{r'}{x'} + \cdots + \frac{r}{x} \times \frac{r'}{x'} + \cdots$$

$$+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots.$$ Q.E.D.

NOTE. Manifestly, if a term in either factor is negative, the corresponding partial product is negative or positive according as the co-factor of this term is positive or negative.

(*b*) *Three or more factors.*

For ∵ the product of two factors is the sum of the partial products of each term of one factor by each term of the other, [(*a*)

and ∵ the product of this product by a third factor is the sum of the partial products of each term of this product by each term of the third factor; [(*a*)

∴ the product of three factors is, etc. Q. E. D.

So, for any number of factors. Q. E. D.

FORM OF PRODUCT.

COR. 1. *The form of a product is independent of the values of the letters that enter into it;* i.e., *the same numerals, letters, exponents, coefficients, and signs, occur and combine in the same order, whatever the numbers for which the letters stand.*

COR. 2. *If each factor be symmetric as to two or more letters, the product is also symmetric as to the same letters.*

COR. 3. *If any values be given to the letters, or if any definite relations be assumed between their values, the value of the product equals the product of the values of the factors.*

COR. 4. *The sum of the coefficients of a product is the continued product of the sum of the coefficients of the first factor, by the sum of the coefficients of the second factor, and so on.*

COR. 5. *The degree of the* $\left\{\begin{matrix}\textit{highest}\\ \textit{lowest}\end{matrix}\right.$ *term of a product, as to any letter or letters, is the sum of the degrees of the* $\left\{\begin{matrix}\textit{highest}\\ \textit{lowest}\end{matrix}\right.$ *terms of the factors, as to the same letter or letters. In particular, the degree of the product is the sum of the degrees of the several factors.*

COR. 6. *If each factor be homogeneous as to any letter or letters, then the product is homogeneous as to the same letter or letters.*

COR. 7. *The whole number of terms in any product, before reduction, is the continued product of the number of terms in the several factors; and the product of two or more polynomials can never be reduced to less than two terms; viz.: the term of highest degree and the term of lowest degree as to any letter or letters.*

COR. 8. *The value of every rational expression whose elements are commensurable numbers is a commensurable number.*

For $\because$ in such an expression the elements enter only as elements of sums, differences, products, quotients, and integral powers,

and $\because$ these results enter only as elements of new sums, etc., and so on,

and $\because$ the sums, etc., of commensurables are commensurables; [th. 3 cr. 3 nt., th. 4 cr.

$\therefore$ the sums, etc., of the elements are commensurables,

$\therefore$ the sums, etc., of these results and the original elements are commensurables, and so on;

$\therefore$ the final result is commensurable. Q. E. D.

§ 8. PROPORTION.

FOUR numbers are *proportional* (in proportion) when the first is such multiple, part, or parts, of the second, as the third is of the fourth; *i.e.*, when the quotient of the first by the second equals the quotient of the third by the fourth.

E.g., if $a:b=c:d$, then a, b, c, d are proportionals, taken in the order given.

A proportion is also written in the forms $a:b::c:d$ and $\frac{a}{b}=\frac{c}{d}$; and it is read: a *is to* b *as* c *is to* d, or *the ratio of* a *to* b *equals the ratio of* c *to* d, or, more briefly, a *to* b *equals* c *to d.*

These quotients are now called *ratios;* the dividends, *antecedents;* the divisors, *consequents;* the first and fourth terms, *extremes;* the second and third terms, *means;* the fourth term, a *fourth proportional* to the other three.

Three numbers are proportional when the quotient of the first by the second equals the quotient of the second by the third. It is a case of four proportionals wherein the two means are the same number. The second number is a *mean proportional* between the first and third, and the third is a *third proportional* to the first and second.

E.g., $a:b=b:c$, wherein b is a mean proportional between a and c, and c is a third proportional to a and b.

Six or more numbers are in *continued proportion* when the first is to the second as the third is to the fourth, as the fifth is to the sixth, and so on.

E.g., $a:b=c:d=e:f=\cdots$.

By aid of Th. 7 (b) this proportion may be written in the form

$$a:c:e:\cdots=b:d:f:\cdots,$$

wherein $a, c, e, \ldots$ are the antecedents, and $b, d, f, \ldots$ the consequents.

THEOR. 6. *If four numbers be proportional, the product of the extremes equals the product of the means; and, conversely, if the product of two numbers equal the product of two others, the four numbers form a proportion, wherein the factors of one product are the extremes and the factors of the other product are the means.*

(a) Let $a:b=c:d$, then will $ad=bc$.

For $\because\ (a:b)\times bd=(c:d)\times bd,$ [ax. 4

$\therefore\ ad=bc.$ Q. E. D. [th. 3 cr. 9

(b) Let $ad=bc$, then will $a:b=c:d$.

For $\because\ ad:bd=bc:bd,$ [ax. 5

$\therefore\ a:b=c:d.$ Q. E. D. [th. 3 cr. 7, cr. 5 nt.

COR. 1. *If four numbers be proportional, either* $\left\{\begin{matrix}\textit{extreme}\\ \textit{mean}\end{matrix}\right.$ *is the quotient of the product of the* $\left\{\begin{matrix}\textit{means}\\ \textit{extremes}\end{matrix}\right.$ *by the other* $\left\{\begin{matrix}\textit{extreme.}\\ \textit{mean.}\end{matrix}\right.$

For, let $a:b=c:d$,

then $\because\ ad=bc,$ [th.

$\therefore\ a=bc:d,\quad b=ad:c,\quad c=ad:b,\quad d=bc:a.$ [ax. 5

COR. 2. *If three numbers be proportional, either extreme equals the quotient of the square of the mean by the other extreme, and the mean equals the square root of the product of the extremes.*

For, let $a:b=b:c$,

then $\because\ ac=b^2,$ [th.

$\therefore\ a=b^2:c,\quad b=\sqrt{ac},\quad c=b^2:a.$ Q. E. D. [ax. 5, 7

NOTE. The equation $ad=bc$ may be resolved into eight different proportions, four of them with a and d for extremes and b and c for means, and four of them with b and c for extremes and a and d for means. The reader may write them out; he will find two of them given in Th. 7 (a, b).

THEOR. 7. *If four numbers be proportional, they are proportional:*

(*a*) *Inversely: the second to the first as the fourth to the third.*

Let $a : b = c : d$, then will $b : a = d : c$.

For $\because\ ad = bc$, [th. 6

$\therefore\ b : a = d : c$. Q. E. D. [th. 6 cv.

(*b*) *Alternately: the first to the third as the second to the fourth.*

Let $a : b = c : d$, then will $a : c = b : d$.

For $\because\ ad = bc$, [th. 6

$\therefore\ a : c = b : d$. Q. E. D. [th. 6 cv.

(*c*) *By addition or subtraction* (composition or division): *the* $\left\{ \begin{matrix} \textit{sum} \\ \textit{remainder} \end{matrix} \right.$ *of the first* $\left\{ \begin{matrix} \textit{and} \\ \textit{less} \end{matrix} \right.$ *the second, to the first or second, as the* $\left\{ \begin{matrix} \textit{sum} \\ \textit{remainder} \end{matrix} \right.$ *of the third* $\left\{ \begin{matrix} \textit{and} \\ \textit{less} \end{matrix} \right.$ *the fourth, to the third or fourth.*

Let $a : b = c : d$, then will

$$a \pm b : a = c \pm d : c, \text{ and } a \pm b : b = c \pm d : d.$$

For $\because\ ad = bc$, [th. 6

$\therefore\ ac \pm bc = ac \pm ad$ and $ad \pm bd = bc \pm bd$, [ax. 2, 3

i.e., $(a \pm b)c = a(c \pm d)$ and $(a \pm b)d = b(c \pm d)$,

$\therefore\ a \pm b : a = c \pm d : c$ and $a \pm b : b = c \pm d : d$. Q. E. D.

(*d*) *By addition and subtraction* (composition and division): *the sum of the first and second to their remainder as the sum of the third and fourth to their remainder.*

Let $a : b = c : d$, then will $a + b : a - b = c + d : c - d$.

For $\because\ a + b : a = c + d : c$ and $a - b : a = c - d : c$, [(*c*)

$\therefore\ a + b : c + d = a : c$ and $a - b : c - d = a : c$, [(*b*)

$\therefore\ a + b : c + d = a - b : c - d$, [ax. 1

$\therefore\ a + b : a - b = c + d : c - d$. Q. E. D. [(*b*)

COR. *Conversely, if four numbers be proportional,* (a) *inversely,* (b) *alternately,* (c) *by addition or subtraction, or* (d) *by addition and subtraction; then is the first to the second as the third to the fourth.*

The reader may prove, by retracing the steps, from conclusion to data, in each of the above demonstrations.

THEOR. 8. *If there be two or more sets of proportionals, the products of their corresponding terms are proportional.*

Let $a:b=c:d,\ \ a':b'=c':d',\ \ a'':b''=c'':d'',\ \cdots,$
then will $aa'a''\cdots : bb'b''\cdots = cc'c''\cdots : dd'd''\cdots.$

For $\because\ ad=bc,\ \ a'd'=b'c',\ \ a''d''=b''c'',\cdots,$ [th. 6

$\therefore\ ad\cdot a'd'\cdot a''d''\cdots = bc\cdot b'c'\cdot b''c''\cdots,$ [ax. 4

$\therefore\ aa'a''\cdots dd'd''\cdots = bb'b'\cdots cc'c''\cdots,$ [th. 3

$\therefore\ aa'a''\cdots : bb'b''\cdots = cc'c''\cdots : dd'd''\cdots.$ Q. E. D. [th. 6 cv.

COR. 1. *If there be two sets of proportionals, the quotients of their corresponding terms are proportional.*

For, let $a:b=c:d$ and $a':b'=c':d'$,

then $\because\ ad=bc$ and $a'd'=b'c'$, [th. 6

$\therefore\ \dfrac{ad}{a'd'}=\dfrac{bc}{b'c'},$ *i.e.*, $\dfrac{a}{a'}\cdot\dfrac{d}{d'}=\dfrac{b}{b'}\cdot\dfrac{c}{c'},$ [ax. 5, th. 3 cr. 1

$\therefore\ \dfrac{a}{a'}:\dfrac{b}{b'}=\dfrac{c}{c'}:\dfrac{d}{d'}.$ Q. E. D. [th. 6 cv.

COR. 2. *If four numbers be proportional, their like integral powers are proportional.*

The reader may write in formula, and prove.

THEOR. 9. *If six or more numbers be in continued proportion, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Let $a:b=c:d=e:f=\cdots,$ then will

$$a+c+e+\cdots : b+d+f+\cdots = a:b=c:d=\cdots.$$

For $\because\ ad=bc,\ \ af=be,\ \ \cdots,$ [th. 6

$\therefore\ ab+ad+af+\cdots=ba+bc+be+\cdots,$ [ax. 2

$\therefore\ a(b+d+f+\cdots)=b(a+c+e+\cdots),$ [th. 4

$\therefore\ a+c+e+\cdots : b+d+f+\cdots = a:b.$ Q.E.D. [th. 6 cv.

COR. 1. *If* $\mathrm{a:b=c:d=e:f}=\cdots,$

then $\mathrm{ha+kc+le}+\cdots : \mathrm{hb+kd+lf}+\cdots = \mathrm{a:b},$

wherein h, k, l, $\cdots$ *are any numbers.*

The reader may state in words, and prove.

COR. 2. *If* $\mathrm{a:b=c:d=e:f}=\cdots,$

then $\mathrm{ha^n+kc^n+le^n}+\cdots : \mathrm{hb^n+kd^n+lf^n}+\cdots = \mathrm{a^n:b^n},$

wherein h, k, l, $\cdots$ *are any numbers and* n *any integer.*

The reader may state in words, and prove.

§ 9. PROCESS OF ADDITION.

PROB. 1. TO ADD TWO OR MORE NUMBERS.

(a) *The numbers like:*

To the common factor prefix the sum of the coefficients. [th. 4

E.g., 10 ft. down + 20 ft. up + 60 ft. up = 70 ft. up,
10 ft. up + 20 ft. down + 60 ft. down = 70 ft. down.

So, $10x - 15x + 20x - 25x + 30x = 60x - 40x = 20x$, [th. 2
$10ay + 20by - 30cy = (10a + 20b - 30c)y$.

(b) *The numbers unlike:*

Write the numbers together, with their proper signs, in any convenient order. [th. 1

E.g., $19xyz - 29mn + 39a - 49$ is irreducible.

So, $10ay + 20by - 30cy$ is usually not reduced, but may be written $(10a + 20b - 30c)y$.

(c) *Some numbers like and some unlike:*

Unite into one sum each set of like numbers, and write these partial sums, together with the remaining terms, in any order.

E.g., $(a^3 + 3a^2b + 3ab^2 + b^3) + (a^3 - 3a^2c + 3ac^2 - c^3)$
$= 2a^3 + 3a^2(b - c) + 3a(b^2 + c^2) + (b^3 - c^3)$.

So, $$3xy + 7xy + \frac{11m}{n}xy - 15x^2 + \frac{a}{b}x^2 + \frac{a}{b}y^2 + \frac{2c}{b}x^2 - \frac{3c}{b}y^2$$
$$= \frac{a - 15b + 2c}{b}x^2 + \frac{11m + 10n}{n}xy + \frac{a - 3c}{b}y^2.$$

§ 10. PROCESS OF SUBTRACTION.

PROB. 2. TO SUBTRACT ONE NUMBER FROM ANOTHER.

To the minuend add the opposite of the subtrahend. [th. 1 cr. 2

E.g., 90 ft. up − 60 ft. up = 90 ft. up + 60 ft. down = 30 ft. up,
60 ft. up − 90 ft. up = 60 ft. up + 90 ft. down = 30 ft. down,
90 ft. up − 60 ft. down = 90 ft. up + 60 ft. up = 150 ft. up;

i.e., $^{+}90 - {}^{+}60 = {}^{+}30$, $^{+}60 - {}^{+}90 = {}^{-}30$, $^{+}90 - {}^{-}60 = {}^{+}150$.

So, $[2a^3 + 3a^2(b - c) + 3a(b^2 + c^2) + (b^3 - c^3)]$
$- [a^3 - 3a^2c + 3ac^2 - c^3] = a^3 + 3a^2b + 3ab^2 + b^3$.

NOTE 1. The opposite of the subtrahend need not be written; but the sign may be changed and the addition made, mentally.

NOTE 2. The definition of subtraction leads to a more direct operation:

E.g., $^+8 - {}^+3 = {}^+5$, $\because$ $^+8 > {}^+3$ by $^+5$,
$^+8 - {}^-3 = {}^+11$, $\because$ $^+8 > {}^-3$ by $^+11$,
$^-8 - {}^+3 = {}^-11$, $\because$ $^-8 < {}^+3$ by $^+11$, *i.e.*, $^-8 > {}^+3$ by $^-11$,
$^-8 - {}^-3 = {}^-5$, $\because$ $^-8 < {}^-3$ by $^+5$, *i.e.*, $^-8 > {}^-3$ by $^-5$.
$^+8 - {}^+10 = {}^-2$, $\because$ $^+8 < {}^+10$ by $^+2$, *i.e.*, $^+8 > {}^+10$ by $^-2$,
$^+8 - {}^-10 = {}^+18$, $\because$ $^+8 > {}^-10$ by $^+18$,
$^-8 - {}^+10 = {}^-18$, $\because$ $^-8 < {}^+10$ by $^+18$, *i.e.*, $^-8 > {}^+10$ by $^-18$,
$^-8 - {}^-10 = {}^+2$, $\because$ $^-8 > {}^-10$ by $^+2$.

§ 11. PROCESS OF MULTIPLICATION.

PROB. 3. TO MULTIPLY ONE NUMBER BY ANOTHER.

(*a*) *A monomial by a monomial:*

To the product of the numerical coefficients annex the several literal factors, each taken as many times as it is found in both multiplicand and multiplier together. [th. 3

Mark the product $\left\{ \begin{matrix} + \\ - \end{matrix} \right.$ *if the factors are taken in* $\left\{ \begin{matrix} \textit{the same} \\ \textit{a contrary} \end{matrix} \right.$ *sense.* [th. 2 cr. 1

E.g., $^+9\,ab^{-3} \times {}^+7\,a^2c^3 = {}^+63\,a^3b^{-3}c^3$,
$^-5\,xy^2z^{-3} \times {}^+7\,x^3yz^{-2} = {}^-35\,x^4y^3z^{-5}$,
$^+9\,a^{-3}b^3 \times {}^-7\,a^2c^{-3} = {}^-63\,a^{-1}b^3c^{-3}$,
$^-5\,xy^2z^{-3} \times {}^-7\,x^{-4}y^2z^3 = {}^+35\,x^{-3}y^4$.

(*b*) *A polynomial by a monomial:*

Multiply each term of the multiplicand by the multiplier; add the partial products. [th. 4

E.g., $(3\,xy^2 + 7\,y^{-3}z^4 - \frac{3}{5}x^{-5}z) \times {}^-\frac{5}{8}xy^{-3}z^5$
$= -\frac{15}{8}x^2y^{-1}z^5 - \frac{35}{8}xy^{-6}z^9 + \frac{3}{8}x^{-4}y^{-3}z^6$.

(*c*) *A polynomial by a polynomial:*

Multiply each term of the multiplicand by each term of the multiplier; add the partial products. [th. 5

E.g., $(a^2 - ab + b^2) \times (a + b) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3$
$= a^3 + b^3$.

NOTE 1. CHECKS: The work is tested by division, [pr. 4 and sometimes by the principles laid down in [th. 5 cr. 2–7].

NOTE 2. ARRANGEMENT: The work is shortened by arranging the terms of both factors, and of the product, according to the powers of some one letter (called the *letter of arrangement*), and by grouping together like partial products.

E.g., $(a^3+3a^2b+3ab^2+b^3)\times(a^2+2ab+b^2)$ [a, let. of ar.
is written

$$\begin{array}{l}
a^3+3a^2b+3ab^2+b^3 \\
a^2+2ab\ \ +b^2 \\
\hline
\end{array}$$

$$\begin{array}{r|r|r|r|r}
a^5+3 & a^4b+3 & a^3b^2+1 & a^2b^3 & \\
+2 & +6 & +6 & +2 & ab^4 \\
 & +1 & +3 & +3 & +b^5 \\
\hline
\end{array}$$

$$=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

NOTE 3. CROSS-MULTIPLICATION: The work is shortened by grouping and adding mentally like partial products, and writing their sum only. In the example of Note 2,

the computer says:	and writes:
$a^3\times a^2$ is a^5	a^5
$3a^2b\times a^2$ is $3a^4b$, $a^3\times 2ab$ is $2a^4b$, whose sum is $5a^4b$,	$5a^4b$
$3ab^2\times a^2$ is $3a^3b^2$, $3a^2b\times 2ab$ is $6a^3b^2$, $a^3\times b^2$ is a^3b^2, whose sum is $10a^3b^2$,	$10a^3b^2$
$b^3\times a^2$ is a^2b^3, $3ab^2\times 2ab$ is $6a^2b^3$, $3a^2b\times b^2$ is $3a^2b^3$, whose sum is $10a^2b^3$,	$10a^2b^3$
$b^3\times 2ab$ is $2ab^4$, $3ab^2\times b^2$ is $3ab^4$, whose sum is $5ab^4$,	$5ab^4$
$b^3\times b^2$ is b^5,	b^5

and the whole product, as above, is

$$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

So, to multiply 384 by 287, product 110208,

the computer says:	and writes:
$4\times 7=28$;	8
2; $8\times 7=56$, 58; $4\times 8=32$, 90;	0
9; $3\times 7=21$, 30; $8\times 8=64$, 94; $4\times 2=8$, 102;	2
10; $3\times 8=24$, 34; $8\times 2=16$, 50;	0
5; $3\times 2=6$, 11;	11

NOTE 4. DETACHED COEFFICIENTS: When both multiplicand and multiplier are *arranged by* some *one letter*, *i.e.*, are such that, after their coefficients are detached, the remaining factors of successive terms will stand in one constant ratio, the work is shortened by the use of these detached coefficients, thus:

Take the terms of both multiplicand and multiplier in such order that, when the coefficients are detached, the remaining factors (letters of arrangement) *of successive terms shall have a constant ratio.*

Write the coefficients, suppressing the letters of arrangement, with 0 *for the coefficient of any term wanting in either series.*

Multiply the coefficients, and add those partial products that pertain to like terms of the final product.

In the final product restore the suppressed factors: in the first term by actual multiplication, and in the other terms by means of the constant ratio.

E.g., $(a^3+3a^2b+3ab^2+b^3)\times(a^2+2ab+b^2)$,
wherein the constant ratio of the literal parts is $b:a$ in both factors, gives

$$\begin{array}{llllll}
1+ & 3+ & 3+ & 1 & & \\
1+ & 2+ & 1 & & & \\
\hline
1+ & 3+ & 3+ & 1 & & \\
 & +2+ & 6+ & 6+ & 2 & \\
 & & +1+ & 3+ & 3+ & 1 \\
\hline
1+ & 5+ & 10+ & 10+ & 5+ & 1\,;
\end{array}$$

and the product, when the letters of arrangement are restored, is $\quad a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$

CHECK: $1+3+3+1=8,\quad 1+2+1=4,$
$8\times4=32\quad$ and $\quad 1+5+10+10+5+1=32.$

So, $\quad 16(x^3+2x^2+4)\times(x-1)+4(x^2-2x+3)\times(x^2-3)$

gives

$$\begin{array}{rrrrr}
1 & 2 & 0 & 4 & \\
1 & {}^-1 & & & \\
\hline
1 & 2 & 0 & 4 & \\
 & {}^-1 & {}^-2 & 0 & {}^-4 \\
\hline
1 & 1 & {}^-2 & 4 & {}^-4 \\
16 & & & & \\
\hline
16 & 16 & {}^-32 & 64 & {}^-64 \\
4 & {}^-8 & 0 & 24 & {}^-36 \\
\hline
20 & 8 & {}^-32 & 88 & {}^-100,
\end{array}
\qquad
\begin{array}{rrrrr}
1 & {}^-2 & 3 & & \\
1 & 0 & {}^-3 & & \\
\hline
1 & {}^-2 & 3 & & \\
 & & {}^-3 & 6 & {}^-9 \\
\hline
1 & {}^-2 & 0 & 6 & {}^-9 \\
4 & & & & \\
\hline
4 & {}^-8 & 0 & 24 & {}^-36
\end{array}$$

$=20x^4+8x^3-32x^2+88x-100.$

So, $(ax - 3a^3x^{\frac{3}{2}} + a^5x^2) \times (b + 3a^2bx^{\frac{1}{2}} + a^4bx)$, [ratio, $a^2x^{\frac{1}{2}}$,

gives

$$\begin{array}{rrrrr} 1 & {}^-3 & 1 & & \\ 1 & 3 & 1 & & \\ \hline 1 & {}^-3 & 1 & & \\ & 3 & {}^-9 & 3 & \\ & & 1 & {}^-3 & 1 \\ \hline 1 & 0 & {}^-7 & 0 & 1, \end{array} = abx - 7a^5bx^2 + a^9bx^3. \quad \text{Q. E. D.}$$

CHECKS: Let $a^2x^{\frac{1}{2}} = \pm 1$; then,

$$\overline{1 \mp 3 + 1} \cdot \overline{1 \pm 3 + 1} = -5$$
$$= 1 - 7 + 1.$$

This method is a familiar one in Arithmetic.

E.g., $1089 \times 237 = 258{,}093$, or

$$\begin{array}{r} 1089 \\ 237 \\ \hline 7623 \\ 3267 \\ 2178 \\ \hline 258093 \end{array} \qquad \begin{array}{r} 1\,\text{th} + 0\,\text{h} + 8\,\text{t} + 9\,\text{u} \\ 2\,\text{h} + 3\,\text{t} + 7\,\text{u} \\ \hline 7\,\text{th} + 6\,\text{h} + 2\,\text{t} + 3\,\text{u} \\ 3\,\text{tth} + 2\,\text{th} + 6\,\text{h} + 7\,\text{t}\phantom{ + 3\,\text{u}} \\ 2\,\text{hth} + 1\,\text{tth} + 7\,\text{th} + 8\,\text{h}\phantom{ + 7\,\text{t} + 3\,\text{u}} \\ \hline 2\,\text{hth} + 5\,\text{tth} + 8\,\text{th} + 0\,\text{h} + 9\,\text{t} + 3\,\text{u} \end{array}$$

The first form is simply a case of detached coefficients, wherein the denominations and the relations of the several numerals are shown by their positions with reference to each other; as, in the last form, they are shown by words and signs.

NOTE 5. TYPE-FORMS: The work is often shortened by the use of certain simple type-forms, which the reader may prove by actual multiplication and then memorize. He may translate them into words and read them as theorems. They are:

1] $(x + a) \cdot (x + b) = x^2 + (a + b)x + ab$,

2] $(a + b) \cdot (a - b) = a^2 - b^2$,

3] $(a + b)^2 = a^2 + 2ab + b^2$,

4] $(a - b)^2 = a^2 - 2ab + b^2$,

5] $(a + b + c + \cdots)^2 = a^2 + b^2 + c^2 + \cdots$
$+ 2(ab + ac + \cdots + bc + \cdots)$,

i.e., $(\Sigma a)^2 = \Sigma a^2 + \Sigma 2ab$,

wherein $\Sigma a \equiv$ the sum of all the terms of the base,

$\Sigma a^2 \equiv$ the sum of all the possible squares,

and $\Sigma 2ab \equiv$ the sum of all the possible double products;

6] $(a - b) \cdot (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$
$= a^n - b^n$, when n is any integer,

7] $(a + b) \cdot (a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + ab^{n-2} - b^{n-1})$
$= a^n - b^n$, when n is any even integer,

8] $(a + b) \cdot (a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - ab^{n-2} + b^{n-1})$
$= a^n + b^n$, when n is any odd integer.

E.g., $[(x^3+y^3)+2\sqrt{x^3y^3}]\cdot[(x^3+y^3)-2\sqrt{x^3y^3}]$

$$=(x^3+y^3)^2-4x^3y^3 \qquad [2$$
$$=x^6+2x^3y^3+y^6-4x^3y^3 \qquad [3$$
$$=x^6-2x^3y^3+y^6$$
$$=(x^3-y^3)^2, \qquad [4$$

wherein $(x^3+y^3+2\sqrt{x^3y^3})\cdot(x^3+y^3-2\sqrt{x^3y^3})$ is the same function of (x^3+y^3) and $2\sqrt{x^3y^3}$ as $(a+b)\cdot(a-b)$, in the type-form, is of a and b; and later, $(x^3+y^3)^2$ and $(x^3-y^3)^2$ are the same functions of x^3 and y^3 as $(a+b)^2$ and $(a-b)^2$ are of a and b.

The advantage of working by type-forms is that most of the details of multiplication are avoided, and the result is reached directly.

NOTE 6. SUBSTITUTION: The work is often shortened by the substitution, during its progress, of a single letter for a less simple expression.

E.g., to multiply $4a^6x^2+9b^4y^2+5-6a^3b^2xy-2a^3x\sqrt{5}$
$-3b^2y\sqrt{5}$ by $2a^3x+3b^2y+\sqrt{5}$.

Let $\text{A}\equiv 2a^3x,\ \text{B}\equiv 3b^2y,\ \text{C}\equiv\sqrt{5},$

then $(\text{A}^2+\text{B}^2+\text{C}^2-\text{AB}-\text{AC}-\text{BC})\times(\text{A}+\text{B}+\text{C})$

$$=\text{A}^3+\text{B}^3+\text{C}^3-3\,\text{ABC}$$
$$=8a^9x^3+27b^6y^3+5\sqrt{5}-18a^3b^2xy\sqrt{5}.$$

NOTE 7. SYMMETRY: The work is often shortened by noting the symmetry of the factors.

E.g., to develop the product

$$(2a+b+c)\cdot(a+2b+c)\cdot(a+b+2c),$$

write the factors in three lines,

$$\begin{array}{r}2a+\ \ b+\ \ c\\ a+2b+\ \ c\\ \underline{a+\ \ b+2c}\end{array}$$

then ∵ the product has the terms $2a\cdot a\cdot a,=2a^3,$ [th. 5

$$2a\cdot a\cdot b+2a\cdot 2b\cdot a+b\cdot a\cdot a,=7a^2b,$$
$$2a\cdot 2b\cdot 2c+b\cdot c\cdot a+c\cdot a\cdot b$$
$$+2a\cdot c\cdot b+b\cdot a\cdot 2c+c\cdot 2b\cdot a,=16abc;$$

and ∵ every term of the product, being entire and of the third degree, is of like form to one of these as to a, b, c;

and ∵ the product is a symmetric function of a, b, c; [th. 5 cr. 2

$\therefore$ it has likewise the following terms, and no others:

$2b^3$, $2c^3$ as well as $2a^3$,

and $7b^2c$, $7c^2a$, $7ab^2$, $7bc^2$, $7ca^2$ as well as $7a^2b$;

$$\therefore\ (2a+b+c)\cdot(a+2b+c)\cdot(a+b+2c)$$
$$= 2a^3+2b^3+2c^3+7a^2b+7b^2c+7c^2a$$
$$+7ab^2+7bc^2+7ca^2+16abc$$
$$\equiv 2\Sigma a^3+7\Sigma a^2b+16abc;$$

wherein Σa^3 $\equiv$ the sum of all the possible cubes,

and $\Sigma a^2b \equiv$ the sum of all the possible products got by taking one letter twice and another letter once.

CHECK: The sum of the coefficients in each factor is 4,
and in the product it is 64, $=4\times4\times4$. [th. 5 cr. 4

So $\because$ of $(2a+b-c)\cdot(-a+2b+c)\cdot(a-b+2c)$,

the terms in a^3, b^3, c^3 have the same coefficient, -2,
those in a^2b, b^2c, c^2a have the same coefficient, 5,
those in ab^2, bc^2, ca^2 have the same coefficient, -1,
and that in abc has the coefficient 2;

$$\therefore\ -2(a^3+b^3+c^3)+5(a^2b+b^2c+c^2a)-(ab^2+bc^2+ca^2)+2abc,$$
$$\equiv -2\Sigma a^3+5\Sigma a^2b-\Sigma ab^2+2abc, \text{ is the product.}$$

CHECK: The sum of the coefficients, when the brackets are removed, is 8, $=(2+1-1)^3$. [th. 5 cr. 4

So, to develop the sum

$$(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2,$$

get by multiplication, or from the type-form for $(a+b+\cdots)^2$, [5

$$(a+b-2c)^2=a^2+b^2+4c^2+2ab-4ca-4bc,$$

write, by symmetry,

$$(b+c-2a)^2=b^2+c^2+4a^2+2bc-4ab-4ca,$$
$$(c+a-2b)^2=c^2+a^2+4b^2+2ca-4bc-4ab,$$

and add; the result is

$$6(a^2+b^2+c^2-bc-ca-ab),\ \equiv 6(\Sigma a^2-\Sigma ab).$$

CHECK: As Σa^2, Σab, each have three terms, their coefficients in the sum of the three products are the sum of coefficients of a^2, $\cdots$, and of ab, $\cdots$, in the product first got; *i.e.*, 6 and -6.

In such symmetric expressions, where three letters are involved, they may be kept advancing in the same order,

abc, *bca*, *cab* or *acb*, *bac*, *cab*   *ab*, *bc*, *ca* or *ac*, *ba*, *cb*,

as if they were points on a circle following one another round and round in the same rotary direction.

So, $(a+b+c)\cdot(x+y+z) + (a+b-c)\cdot(x+y-z)$
$+ (a-b+c)\cdot(x-y+z) + (-a+b+c)\cdot(-x+y+z)$

$$\begin{array}{l}
= + ax + ay + az + bx + by + bz + cx + cy + cz \\
\;\;\, + \quad + \quad - \quad + \quad + \quad - \quad - \quad - \quad + \\
\;\;\, + \quad - \quad + \quad - \quad + \quad - \quad + \quad - \quad + \\
\;\;\, + \quad - \quad - \quad - \quad + \quad + \quad - \quad + \quad + \\
= 4\,ax \qquad\qquad\qquad + 4\,by \qquad\qquad\qquad + 4\,cz
\end{array}$$

$\equiv 4\,\Sigma\,ax$, a symmetric expression as to ax, by, and cz.

NOTE 8. CONTRACTION: When only the first few terms of a product are wanted, the work may be shortened by omitting all partial products that do not enter into the required terms.

E.g., to develop $(1-3x+5x^2-\cdots)^2$ as far as the term in x^2:

$$\begin{array}{rrr}
1-3x+\;5x^2-\cdots & \text{or} & 1-3+\;5-\cdots \\
\underline{1-3x+\;5x^2-\cdots} & & \underline{1-3+\;5-\cdots} \\
1-3x+\;5x^2 & & 1-3+\;5 \\
-3x+\;9x^2 & & -3+\;9 \\
\underline{+\;5x^2} & & \underline{+\;5} \\
1-6x+19x^2 & & 1-6+19
\end{array}$$

So, to find the product, omitting x^4 and higher powers, of
$(1+x+x^2+\cdots)\cdot(1-2x+3x^2-\cdots)\cdot(1+4x+9x^2+\cdots)$,

write

$$\begin{array}{rrrrcrrrr}
\overline{1} & \overline{1} & \overline{1} & \overline{1} & \times & \overline{1} & \overline{^-2} & \overline{3} & \overline{^-4} \\
 & ^-2 & ^-2 & ^-2 & & & & & \\
 & & 3 & 3 & & & & & \\
 & & & ^-4 & & & & & \\
\hline
1 & ^-1 & 2 & ^-2 & \times & \overline{1} & \overline{4} & \overline{9} & \overline{16} \\
 & 4 & ^-4 & 8 & & & & & \\
 & & 9 & ^-9 & & & & & \\
 & & & 16 & & & & & \\
\hline
1 & 3 & 7 & 13, & & & & &
\end{array}$$

and the product, as far as wanted, is

$$1+3x+7x^2+13x^3.$$

If in this example the value of x be small, say $x < .01$, the part product found above approximates very closely to the value of the true product, and for many purposes may be used in place of it. The work is further shortened by cross-multiplication.

This method of contracted multiplication may be used, with great profit, with decimal fractions.

E.g., to find the product 37.8562×14.9716, correct to two places, and $.2819 \times .3781 \times .2148$ to three places.

$$
\begin{array}{r}
37.8562 \\
\underline{14.9716} \\
378.562 \\
151.425 \\
34.070 \\
2.650 \\
38 \\
\underline{23} \\
566.77
\end{array}
\qquad \text{and} \qquad
\begin{array}{r}
.2819 \\
\underline{.3782} \\
.0846 \\
197 \\
\underline{22} \\
.107
\end{array}
\qquad\qquad
\begin{array}{r}
.107 \\
\underline{.2148} \\
.0214 \\
11 \\
\underline{5} \\
.023
\end{array}
$$

In writing down the partial products, carry what would have been carried had the multiplication been made in full.

E.g., the partial product $23 = 3 \times 6 + 5$, carried from 8×6.

§ 12. PROCESS OF DIVISION.

PROB. 4. TO DIVIDE ONE NUMBER BY ANOTHER.

(*a*) *A monomial by a monomial:*

To the quotient of the numerical coefficients annex the several literal factors, each taken as many times as its exponent in the dividend exceeds its exponent in the divisor. [th. 3 cr. 9

Mark the quotient $\left\{ \begin{matrix} + \\ - \end{matrix} \right.$ *if the terms are of* $\left\{ \begin{matrix} \textit{the same} \\ \textit{a contrary} \end{matrix} \right.$ *sense.*

E.g.,

$$
\begin{aligned}
63\,a^{-2}b^2d^5 &: 7\,ac^3d^5 = 9\,a^{-3}b^2c^{-3}, \\
-35\,x^4y^{-4}z^4 &: 5\,xy^{-2}z^5 = -7\,x^3y^{-2}z^{-1}, \\
\frac{63}{25}a^{-2}b^{-2}d^{-5} &: -\frac{7}{5}ac^{-3}d^{-5} = -\frac{9}{5}a^{-3}b^{-2}c^3, \\
-\frac{2}{3}x^{-4}y^3z^{-3} &: -\frac{7}{2}x^{-5}y^{-2}z^3 = \frac{4}{21}xy^5z^{-6}.
\end{aligned}
$$

(*b*) *A polynomial by a monomial:*

Divide each term of the dividend by the divisor; add the partial quotients. [th. 4

E.g.,

$$
\begin{aligned}
&\left(45\,x^3y^2z + \frac{105}{11}xz^{-1} - \frac{165}{14}x^{-2}y^{-3}z^{-4}\right) : -\frac{15}{7}xy^2z^{-2} \\
&= -21\,x^2z^3 - \frac{49}{11}y^{-2}z + \frac{11}{2}x^{-3}y^{-5}z^{-2}.
\end{aligned}
$$

(c) *Any number by a polynomial:*

Arrange the terms of both dividend and divisor according to the powers of some one letter, (preferably the letter whose powers are most numerous, and the highest powers first).

Divide the first term of the dividend by the first term of the divisor, called the TRIAL DIVISOR.

Multiply the whole divisor by this partial quotient, and subtract the product from the dividend.

Repeat the work upon the remainder as a new dividend, and so on, till the whole dividend is exhausted, or till the requirements of the work are satisfied.

Add the partial quotients: their sum is the quotient, the part of the dividend left undivided is the remainder, and the sum of the quotient and the fraction $\dfrac{\textit{remainder}}{\textit{divisor}}$ *is the* COMPLETE QUOTIENT.

E.g., to divide a^3+b^3 by $a+b$.

$$\begin{array}{r|l} a^3+b^3 & a+b \\ \cline{2-2} a^3+a^2b & a^2-ab+b^2 \\ \hline -a^2b+b^3 & \\ -a^2b-ab^2 & \\ \cline{1-1} ab^2+b^3 & \\ ab^2+b^3 & \\ \cline{1-1} \end{array} \qquad \text{or} \qquad \begin{array}{ll} a^2-ab+b^2 & \textit{quotient,} \\ a+b & \textit{divisor,} \\ \underline{a^3+b^3} & \textit{dividend.} \\ \underline{-a^2b} & \\ \underline{ab^2} & \end{array}$$

The first is the ordinary form for division; and the other is an abbreviated form, in which the partial products are not written, but the partial remainders only. The relative position of dividend, divisor, and quotient, is unimportant except as a matter of convenience.

So $(ar^n-a) : (r-1) = ar^{n-1}+ar^{n-2}+ar^{n-3}+\cdots+ar+a$,
and $(a-ar^n) : (1-r) = a+ar+ar^2+\cdots+ar^{n-2}+ar^{n-1}$.

So
$$\begin{array}{rr|l||l} x^3+2ax^2+a^2 & & x+2a^2b & \left.\begin{array}{r} x\;+a \\ -b \end{array}\right| \textit{divisor,} \\ \textit{dividend} & \begin{array}{r} +ab \\ +b^2 \end{array} & & \\ \cline{4-4} & & & \left.\begin{array}{r} x^2+a \\ +b \end{array}\right| \left.\begin{array}{r} x+ab \\ +2b^2 \end{array}\right| \textit{quotient,} \\ \hline \end{array}$$
$$\left.\begin{array}{r} a \\ +b \end{array}\right| \left.\begin{array}{r} x^2 \quad ab \\ ,+2b^2 \end{array}\right| \left.\begin{array}{r} x \qquad a^2b \\ ,\quad -ab^2 \\ +2b^3 \end{array}\right| \textit{remainder.}$$

This example is solved by the second method, and the several

partial remainders are written on one horizontal line simply as a matter of convenience. The complete quotient is

$$\left.\begin{array}{r} x^2 + a \\ + b \end{array}\right| \left.\begin{array}{r} x + ab \\ + 2b^2 \end{array}\right| + \frac{a^2 b - ab^2 + 2b^3}{x + a - b}.$$

NOTE 1. CHECKS: The work is tested by multiplying together the divisor and quotient, and adding in the remainder. In this process the order of multiplication, quotient $\times$ divisor, is preferable, as least like the order of the work of division.

If the division be exact, principles laid down in [th. 5 crs. 2–7] are also useful.

Another test is to reproduce the divisor by subtracting the remainder from the dividend and dividing by the quotient.

NOTE 2. DIFFERENT FORMS OF QUOTIENT. Unless the division is *exact*, *i.e.*, leaves no remainder, the quotient and remainder are commonly different for every different choice of trial divisor that may be made.

E.g., $(x^2 + 1) : (x + 1)$ gives for quotient and remainder:

$x - 1,\ 2$ when x is trial divisor,

$1 - x,\ 2x^2$ when 1 is trial divisor.

So $(x^2 + y^2 + z^2) : (x + y + z)$ gives for quotient and remainder:

$x - y - z,\ 2(y^2 + yz + z^2)$ when x is trial divisor,

$y - z - x,\ 2(z^2 + zx + x^2)$ when y is trial divisor,

$z - x - y,\ 2(x^2 + xy + y^2)$ when z is trial divisor.

The quotient and remainder depend also upon the extent to which the division is carried.

E.g., $(x^2 + 1) : (x + 1)$ gives for quotient and remainder:

	$x,$	$-x + 1$	
or	$x - 1,$	2	
or	$x - 1 + 2x^{-1},$	$-2x^{-1}$	
or	$\cdots\ \cdots$	$\cdots\ \cdots$	

$$\begin{array}{r|l} x^2 + 1 & x + 1 \\ \hline -x + 1 & x - 1 + 2x^{-1} - 2x^{-2} \\ \hline 2 & \\ \hline -2x^{-1},\ 2x^{-2},\ \cdots & \end{array}$$

But the complete quotient is the same in value, whatever the trial divisor and the extent of the division.

E.g., $$x + \frac{-x+1}{x+1} = x - 1 + \frac{2}{x+1} = x - 1 + 2x^{-1} + \frac{-2x^{-1}}{x+1} = \cdots$$

$$= 1 + \frac{-1+x}{1+x} = 1 - x + \frac{2x^2}{1+x} = \cdots,\ \text{when } 1 + x \text{ is divisor.}$$

NOTE 3. DETACHED COEFFICIENTS: When both dividend and divisor are *arranged by* some *one letter*, *i.e.*, are such that, after the coefficients are detached, the remaining factors of successive terms will stand in one constant ratio, the work is shortened by the use of these detached coefficients, thus:

Take the terms of both dividend and divisor in such order that, when the coefficients are detached, the remaining factors of successive terms shall have a constant ratio.

Write the coefficients, suppressing the letters of arrangement, with 0 *for the coefficient of any term wanting in either series.*

Divide, treating the set of coefficients in the dividend, and that in the divisor, as polynomials.

In the quotient restore the suppressed factors: in the first term, by actual division, and in the successive terms, by means of the constant ratio.

E.g., $(a^3+b^3):(a+b)$ and $(a^3-b^3):(a+b)$

give

$$\begin{array}{l|l} 1\ \ 0\ \ 0\ \ 1 & 1\ \ 1 \\ \underline{1\ \ 1} & 1\ \ {}^-1\ \ 1 \\ \ \ {}^-1\ \ 0\ \ 1 & \\ \ \ \underline{{}^-1\ \ {}^-1} & \\ \ \ \ \ \ \ 1\ \ 1 & \end{array} \qquad \text{and} \qquad \begin{array}{r|l} \underline{1\ \ 0\ \ 0\ \ {}^-1} & 1\ \ 1 \\ {}^-1,\ 1,\ {}^-2 & 1\ \ {}^-1\ \ 1 \end{array}$$

and the quotients and remainder are

$$a^2-ab+b^2 \text{ and } a^2-ab+b^2,\ -2b^3.$$

So $(x^4-2x^3+6x-9):(x^2-3)$

gives

$$\begin{array}{l|l} 1\ \ {}^-2\ \ 0\ \ 6\ \ {}^-9 & 1\ \ 0\ \ {}^-3 \\ \ \ \underline{{}^-2\ \ 3} & 1\ \ {}^-2\ \ 3 \\ \ \ \ \ \ \ \underline{3\ \ 0} & \end{array}$$

and the exact quotient is

$$x^2-2x+3.$$

NOTE 4. SYNTHETIC DIVISION: When the first coefficient of the divisor is 1, the work by detached coefficients is further shortened, thus:

Suppress the first coefficient of the divisor, and replace the other coefficients by their opposites, so that the partial products may be added; write the skeleton divisor thus changed preferably at the left in a vertical column running down, the partial products under

the dividend, and the quotient under these partial products; add the partial products as needed.

E.g., $(x^4+3x^3+3x^2+2):(x^2-2x+3)$

gives

$$\begin{array}{r|rrrrr} & 1 & 3 & 3 & 0 & 2 \\ 2 & & 2 & 10 & 20 & \\ ^-3 & & & ^-3 & ^-15 & ^-30 \\ \hline & 1 & 5 & 10, & 5 & ^-28 \end{array}$$

and the quotient and remainder are

$$x^2+5x+10, \text{ and } 5x-28.$$

NOTE 5. TYPE-FORMS: If the dividend and divisor can be reduced to known type-forms, *e.g.* [1–8], then the quotient may be written directly.

E.g., $(x^2+7x+12):(x+4)=x+3,$ [1

$(x^2-y^2):(x+y)=x-y,$ [2

$(x^8-y^8):(x^2-y^2)=x^6+x^4y^2+x^2y^4+y^6,$ [6

wherein $x^2\equiv a,\ y^2\equiv b,\ 4\equiv n,$ of the type-form.

NOTE 6. SUBSTITUTION: The work is often shortened by the substitution, during its progress, of a single letter for a less simple expression.

E.g., $(8a^9x^3+27b^6y^3+5\sqrt{5}-18a^3b^2xy\sqrt{5})$

$:(2a^3x+3b^2y+\sqrt{5})$

when $\text{A}\equiv 2a^3x,\ \text{B}\equiv 3b^2y,\ \text{C}\equiv\sqrt{5},$

gives $(\text{A}^3+\text{B}^3+\text{C}^3-3\text{ABC}):(\text{A}+\text{B}+\text{C})$

$=\text{A}^2+\text{B}^2+\text{C}^2-\text{AB}-\text{AC}-\text{BC}$

$=4a^6x^2+9b^4y^2+5-6a^3b^2xy-2a^3x\sqrt{5}-3b^2y\sqrt{5}.$

NOTE 7. SYMMETRY: If the dividend and divisor be both symmetric as to two or more letters, and if there be no remainder, then the quotient will be a symmetric function of the same letters. It is then often sufficient to get a few characteristic terms, and to write the rest therefrom by symmetry.

E.g., to divide $(a^2+b^2+c^2+2ab+2bc+2ca)$ by $(a+b+c)$, wherein both elements are symmetric functions of a, b, c.

Manifestly a is the first term of the quotient,

$\therefore$ b and c are also probable terms of the quotient;

and $\because$ the product $(a+b+c)\cdot(a+b+c)$ is the given dividend,

$\therefore$ the division is complete, and $a+b+c$ is the quotient.

So $(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2+3xyz):(x+y+z)$,
a quotient of symmetric functions of x, y, and z,
gives xy for one term of the quotient;
$\therefore$ yz and zx are also probable terms of the quotient;
and $\because$ the product $(x+y+z)(xy+yz+zx)$ is the given dividend,
$\therefore$ the division is complete, and $xy+yz+zx$ is the quotient sought.

Had the last term of the dividend been $4xyz$, or any other number except $3xyz$, there would have been a remainder. The reader must therefore use great caution if he employs "symmetry" in division. He may safely use it as suggestive of the true answer, but hardly ever as conclusive.

NOTE 8. CONTRACTION: When only the first few terms of a quotient are wanted, the work is shortened by omitting all partial products that do not affect the required terms.

E.g., $(1+x+x^2+x^3+\cdots):(1-2x+3x^2-4x^3+\cdots)$
to four terms

gives
$$\begin{array}{r|rrrr}
 & 1 & 1 & 1 & 1 \\
2 & & 2 & 6 & 8 \\
{}^-3 & & & {}^-3 & {}^-9 \\
4 & & & & 4 \\
\hline
1 & 1 & 3 & 4 & 4,
\end{array}$$

and the quotient, as far as wanted, is
$$1+3x+4x^2+4x^3.$$

§ 13. OPERATIONS ON FRACTIONS.

PROB. 5. TO REDUCE A FRACTION TO LOWER TERMS.

Divide both terms by any same number that divides them without a remainder; the quotients are the terms of the reduced fraction.

[th. 3 cr. 7, cr. 5 nt.

E.g., $\dfrac{36a^4b^2c^3}{24a^5bx}=\dfrac{3bc^3}{2ax}$,

wherein the divisor is $12a^4b$; and this common operator may be written under the sign $=$, so that the whole stands in the form
$$\frac{36a^4b^2c^3}{24a^5bx}\underset{(12a^4b)}{=}\frac{3bc^3}{2ax}.$$

NOTE. For reduction to lowest terms, see

PROB. 6. TO REDUCE A FRACTION TO A GIVEN NEW DENOMINATOR OR NUMERATOR.

Divide the new denominator or numerator by the old, and multiply both terms of the fraction by the quotient.

[th. 3 cr. 7, cr. 5 nt.

E.g., to reduce $\frac{3x^2y}{2a^2b}$ to an equivalent fraction whose denominator is a^3bc:

$\because \; a^3bc : 2a^2b = \frac{1}{2}ac,$

$\therefore \; \frac{3x^2y}{2a^2b} \underset{(\frac{1}{2}ac)}{=} \frac{\frac{3}{2}acx^2y}{a^3bc}.$

So to reduce $\frac{2x^2z}{3a^2c}$ to an equivalent fraction whose numerator is $6x^2yz$:

$\because \; 6x^2yz : 2x^2z = 3y,$

$\therefore \; \frac{2x^2z}{3a^2c} \underset{(3y)}{=} \frac{6x^2yz}{9a^2cy}.$

NOTE. By this rule any entire or mixed number is reduced to a simple fraction.

E.g., $x + 2a = \frac{x+2a}{1} = \frac{dx + 2ad}{d}.$

So $x + 2a + \frac{a^2}{x} = \frac{x^2 + 2ax + a^2}{x}.$

PROB. 7. TO REDUCE TWO OR MORE FRACTIONS TO A COMMON DENOMINATOR.

Over the continued product of the denominators, write the product of each numerator into all the denominators except its own.

[th. 3 cr. 7, cr. 5 nt.

E.g., $\frac{5xy}{2a}, \frac{3bc}{x}, \frac{3(a-b)}{7} = \frac{35x^2y}{14ax}, \frac{42abc}{14ax}, \frac{6ax(a-b)}{14ax}.$

NOTE. The fractions may be reduced by Pr. 6 to any common denominator whatever; but this usually leads to complex fractions, which the rule of Pr. 7 avoids when the given fractions are simple.

For reduction to lowest common denominator, see III. § 6.

PROB. 8. TO ADD FRACTIONS.

Reduce the several fractions to a common denominator, and write the sum of the new numerators over the common denominator.

$$E.g.,\ \frac{3\,bc^3}{2\,ax}+\frac{3\,(a-b)}{7}=\frac{21\,bc^3+6\,ax\,(a-b)}{14\,ax}.$$

NOTE. Subtraction is but a case of addition; *add the opposite of the subtrahend.*

$$E.g.,\ \frac{3\,bc^3}{2\,ax}-\frac{3\,(a-b)}{7}=\frac{21\,bc^3-6\,ax\,(a-b)}{14\,ax}.$$

PROB. 9. TO MULTIPLY FRACTIONS.

Write the product of the numerators over the product of the denominators. [th. 3 cr. 1, 3

$$E.g.,\ \frac{3\,bc^3}{2\,ax}\times\frac{3\,(a-b)}{7}=\frac{9\,bc^3\,(a-b)}{14\,ax}.$$

NOTE. Division is but a case of multiplication; *multiply by the reciprocal of the divisor.*

$$E.g.,\ \frac{3\,bc^3}{2\,ax}:\frac{3\,(a-b)}{7}=\frac{3\,bc^3}{2\,ax}\times\frac{7}{3(a-b)}=\frac{7\,bc^3}{2\,ax(a-b)}.$$

§ 14. EXAMPLES.

§§ 9, 10. PROBS. 1, 2.

1–8. Free from brackets and reduce to simplest form:
(*a*) removing first the inner brackets, and proceeding outwards;
(*b*) removing first the outer brackets, and proceeding inwards;
(*c*) freeing together all terms of a kind, from all the brackets.

1. $a-[b-(c-d)]$.
2. $a-\{a+b-[a+b-c-(a-b+c)]\}$.
3. $-\{(1+2x+9x^2)+[(3+2x-x^2)$
$\quad -(2+5x+7x^2)+(-3+3x-2x^2)]\}$.
4. $\frac{1}{3}[\frac{3}{2}(x-a)+(y-b)]+\frac{1}{3}[2(x-a)+\frac{1}{2}(a-x)+2(y-b)]$.
5. $-\frac{1}{8}\{[(5a-4b+3c)-(-3a+4b-c)]$
$\quad -[(6a-8c)-(a-b+9c)]\}$.
6. $\frac{1}{2}(ax^2+bx+c)-\frac{1}{4}[(ax^2-bx+c)+\frac{1}{2}(\frac{1}{2}ax^2+bx-\frac{1}{2}c)$
$\quad +\frac{1}{4}(-ax^2+2bx+c)]$.

7. $\frac{3}{2}[(x^2-y^2)-(x^2+2xy+y^2)]-[(2xy-x^2-y^2)-\frac{1}{2}(x^2+y^2)]$.

8. $1.25[1.12x-.24(x-.5)]$
$\quad -\frac{1}{3}[.21(x+1)-.15(1-.16x)-.12(x-\overline{1-.25x})]$.

9. Add $(a-2p)x^3+(q-b)x^2+(3c-2r)x+(3p-a)x^3$
$\quad -(2x^2-x)-(c-1)x-(b+q)x^2-(p-a)x^3$
$\quad -x^3+3bx^2-(c-2r)x$,
and arrange the sum to ascending powers of x.

10. Arrange $a^3+b^3+c^3+3a^2b+3b^2c+3c^2a+3ab^2+3bc^2$
$\quad +3ca^2+6abc$
(*a*) to ascending powers of a, using vertical bars,
(*b*) to ascending powers of b, using horizontal bars,
(*c*) to ascending powers of c, using brackets.

11. Add $x^4+3xy^3-xz^3+x^3y+x^3z+3x^2y^2+3x^2z^2+3xy^2z$
$\quad -3xyz^2-6x^2yz-x^3y+y^4-yz^3-3x^2y^2-3xy^2z$
$\quad -3xy^3-3xyz^2-3y^3z+3y^2z^2-6xy^2z-x^3z+3y^3z$
$\quad +z^4+3x^2yz-3x^2z^2+3xy^2z+xz^3-3y^2z^2+yz^3+3xyz^2$,
and arrange the sum to descending powers of x, and the coefficients to descending powers of y.

12. From $a^5-4a^3b^2-8a^2b^3-17ab^4-12b^5$, subtract successively
$\quad a^5-2a^4b-3a^3b^2,\ 2a^4b-4a^3b^2-6a^2b^3,$
$\quad 3a^3b^2-6a^2b^3-9ab^4$, and $4a^2b^3-8ab^4-12b^5$.

13. If $\mathrm{s}\equiv(a+b+c)x+(a+b+c)y$, $\mathrm{u}\equiv(b+c)x+(2b-c)y$,
$\quad \mathrm{v}\equiv(c+a)x+(2c-a)y$, and $\mathrm{w}\equiv(a+b)x+(2a-b)y$;
find the values of $(\mathrm{s}-\mathrm{u})+(\mathrm{v}-\mathrm{w})$, $(\mathrm{s}-\mathrm{v})+(\mathrm{w}-\mathrm{u})$, and $(\mathrm{s}-\mathrm{w})+(\mathrm{u}-\mathrm{v})$, and the sum of these three sums.

14. Express by brackets, each preceded by $+$; each, by $-$; each beginning with a $+$ term; taking the terms
(*a*) two together, in their order,
(*b*) three together, with an inner bracket embracing the last two of each triplet:
$\quad -3c+4d-2e+3f+2a-5b;$
$\quad -2e+3f+2a-5b-3c-4d;$
$\quad 2a-5b-3c-4d-2e+3f;$
$\quad a+b+c-a-b+c+a-b-c-a+b-c;$
$\quad abc-abd+abe-acd+ace-bcd+bce-bde+ade.$

15–79. Multiply and divide as shown by the signs; use the methods given in the problems and notes specified:

§ 11. PROB. 3, and NOTE 3.

15. $(x^3-2x^2-3x+1)\cdot(2x^2-3x+4)$.

16. $(x^2+y^2+z^2+xy+yz-zx)\cdot(x-y+z)$.

17. $(x^2+ax-b^2)\cdot(x^2+bx-a^2)\cdot(x-\overline{a+b})$.

18. $[(a-1)x^3+(a-1)^2x^2+(a-1)^3x]\cdot[(a+1)x+(a+1)^2+(a+1)^3x^{-1}]$.

Use vertical bars to join coefficients of like powers of x in the product.

19. $(a^m+3b^n-2c^p)\cdot(a^{-m}-3b^{-n}+2c^{-p})$.

20. $\overline{x^2+\begin{array}{r|}a\\+b\end{array}\,x+ab}\cdot\overline{x^2+\begin{array}{r|}c\\+d\end{array}\,x+cd}$; $\overline{x^2-\begin{array}{r|}a\\-b\end{array}\,x+ab}\cdot\overline{x^2+\begin{array}{r|}c\\+d\end{array}\,x+cd}$.

21. $\overline{x^2+\begin{array}{r|}a\\+b\end{array}\,x+ab}\cdot\overline{x^2-\begin{array}{r|}a\\-b\end{array}\,x+ab}$; $\overline{x^2-\begin{array}{r|}a\\-b\end{array}\,x+ab}\cdot\overline{x^2-\begin{array}{r|}c\\-d\end{array}\,x+cd}$.

22. $(x+a)\cdot(x+b)\cdot(x+c)\cdot(x+d)$, at one operation.

§ 11. PROB. 3, NOTE 4.

23. $(x^6-3x^4y^2+3x^2y^4-y^6)\cdot(x^8-4x^6y^2+6x^4y^4-4x^2y^6+y^8)$.

24. $(x^3-2x^2+1)\cdot(2x^2-3x+4)\cdot(x+1)$.

25. $(x^2-mx+m^2)\cdot(x^2+mx+m^2)\cdot(x^4+m^2x^2+m^4)$.

26. Show that $x\cdot(x+1)\cdot(x+2)\cdot(x+3)+1=(x^2+3x+1)^2$.

27. Show that $(y-1)\cdot y\cdot(y+1)\cdot(y+2)+1=(y^2+y-1)^2$.

28. What function must x be of y so that Exs. 26 and 27 shall be precisely the same equation?

29. $(ay^6+by^4z^3-cy^2z^6)\cdot(ay^5z^2-by^3z^5+cyz^8)$.

30. $(2x+3)\cdot(3x-4)$; $(3y-5)\cdot(2y+7)$.

31. $(x^2+3x+2)\cdot(x^2-3x+2)$; $(2-4y^2)\cdot(1+2y^2)$.

32. $(x^3+3x^2y+3xy^2+y^3)\cdot(x^2+2xy+y^2)$;
$(2x^3-3x^2y+2y^3)\cdot(2x^3+3xy^2+2y^3)$.

33. $(2x-5)^2$; $(y+2y^2+3y^3)^2$; $(2-3z-3z^2+2z^3)^2$.

34. 13×15; 35×79; 234×432; 135.7×12.34.

35. 18^2; 37^2; 109^2; 163^2; 725^2; 1881^2; 70.23^2.

§ 11. PROB. 3, NOTE 5.

36. $(x+2)\cdot(x+3)$; $(x+2)\cdot(x-3)$; $(x-2)\cdot(x+3)$; $(x-2)\cdot(x-3)$.

37. $(y+a)\cdot(y+b)$; $(y-a)\cdot(y-b)$; $(y-a)\cdot(y+b)$; $(y+a)\cdot(y-b)$.

38. $(x+\overline{a+b})\cdot(x+\overline{c+d})$; $(x-\overline{a+b})\cdot(x-\overline{c+d})$.

39. $(3a^2x^3+5b^3c^2)\cdot(3a^2x^3+5b^2c^3)$.

40. $(ax^2+y^3)\cdot(ax^2-y^3)$; $(2x^3+3y^2z)\cdot(2x^3-3y^2z)$.

41. $(m^2z^{\frac{1}{2}}-n^2y^{\frac{1}{2}})\cdot(m^2z^{\frac{1}{2}}+n^2y^{\frac{1}{2}})$; $(2+\sqrt{3})\cdot(2-\sqrt{3})$.

42. $(x-a)\cdot(x+a)$; $(x^2-a^2)\cdot(x^2+a^2)$; $\cdots$; $(x^n-a^n)\cdot(x^n+a^n)$.

43. $(1-x)\cdot(1+x)\cdot(1+x^2)\cdot(1+x^4)\cdot(1+x^8)\cdots(1+x^{2^n})$.

44. $(x+3y)^2$; $(x-3y)^2$; $(x\pm 3)^2$; $(2x^2\pm 3y^2)^2$.

45. $(a+\overline{b-c})^2$; $(a-\overline{b+c})^2$; $(\overline{a+b}\pm\overline{c-d})^2$.

46. $(x+y+z)^2$; $(2x+3y-4z)^2$; $(xy+yz+zx)^2$.

47. $(a+b+c)\cdot(-a+b+c)\cdot(a-b+c)\cdot(a+b-c)$.

48. $(a+2b-3c-d)\cdot(a-2b+3c-d)$.

49. $(a+mx-nx^2)\cdot(a-mx+nx^2)$.

50. $\left(x+\dfrac{x^2}{2a}-\dfrac{x^3}{4a^2}\right)\cdot\left(1-\dfrac{x}{2a}+\dfrac{x^2}{4a^2}\right)$.

51. $(a^2+ax+x^2)\cdot(a-x)\times(a^2-ax+x^2)\cdot(a+x)$.

52. $(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+xy^{n-2}+y^{n-1})\cdot(x-y)$.

53. $(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\cdots\pm xy^{n-2}\mp y^{n-1})\cdot(x+y)$.

54. $(p+pr+pr^2+pr^3+\cdots+pr^{n-1})\cdot(r-1)$.

§ 11. PROB. 3, NOTE 6.

55. $(\overline{u+v}+\overline{x+y})^2+(\overline{u+v}-\overline{x+y})^2$.

56. $(\overline{u-v}+\overline{x-y})^2-(\overline{u-v}-\overline{x-y})^2$.

57. $(\overline{a+b}^2+3\,\overline{a+b}\cdot\overline{m+n}+\overline{m+n}^2)\cdot(\overline{a+b}^2-3\,\overline{a+b}\cdot\overline{m+n}$
$+\overline{m+n}^2)$

58. $(\overline{a+b}+3\,\overline{a+b}^{\frac{1}{2}}\cdot\overline{m+n}^{\frac{1}{2}}+\overline{m+n})\cdot(\overline{a+b}-3\,\overline{a+b}^{\frac{1}{2}}\cdot\overline{m+n}^{\frac{1}{2}}$
$+\overline{m+n})$.

59. $(\sqrt{2}+\sqrt{3}+\sqrt{5})\cdot(-\sqrt{2}+\sqrt{3}+\sqrt{5})\cdot(\sqrt{2}-\sqrt{3}+\sqrt{5})$
$\cdot(\sqrt{2}+\sqrt{3}-\sqrt{5})$.

§ 11. PROB. 3, NOTE 7.

60. $(ab+cd)^2+(bc+ad)^2+(ca+bd)^2=\Sigma a^2b^2+6\,abcd.$

61. $(ax-by)^2+(bx-cy)^2+(cx-ay)^2$
$\qquad=(ay-bx)^2+(by-cx)^2+(cy-ax)^2.$

62. $(-x+y+z)(x-y+z)(x+y-z)=\Sigma x^2y-\Sigma x^3.$

63. $(a+b)\cdot(c+d)+(a+b)\cdot(c-d)+(a-b)\cdot(c+d)$
$\qquad+(a-b)\cdot(c-d).$

64. $(ax+by)\cdot(bx+ay)+(ax-by)\cdot(bx-ay).$

65. $(x+y+z)^3-(x^3+y^3+z^3)=3(x+y)\cdot(y+z)\cdot(z+x).$

66. $(a+b+c)^3=\Sigma a^3+3\,\Sigma a^2b+6\,abc.$

67. If $\Sigma a\equiv a+b+c+\cdots,$
then $(\Sigma a)^3\ =\Sigma a^3+\Sigma a^2b+6\,\Sigma abc,$
$\Sigma a\cdot\Sigma a^2=\Sigma a^3+\Sigma a^2b,$
$\Sigma a\cdot\Sigma ab=\Sigma a^2b+3\,\Sigma abc.$

68. $(-a+b+c+d)\cdot(a-b+c+d)\cdot(a+b-c+d)\cdot(a+b$
$\qquad+c-d)=-\Sigma a^4+2\,\Sigma a^2b^2+8\,abcd.$

69. $(ax+by+cz)\cdot(bx+cy+az)\cdot(cx+ay+bz)$
$\qquad=abc(x^3+y^3+z^3)+(a^3+b^3+c^3)\,xyz+3\,abc\cdot xyz$
$\qquad+(ab^2+bc^2+ca^2)\ (xy^2+yz^2+zx^2)$
$\qquad+(a^2b+b^2c+c^2a)\ (x^2y+y^2z+z^2x).$

Test the result severally by [th. 5 crs. 2–7].

§ 11. PROB. 3, NOTE 8.

70. $(1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{3}{4}x^2+\cdots)^2$ to four terms.

71. $(1+.167\,x+.014\,x^2+.001\,x^3)^2$
$\qquad\times(1-.333\,x+.056\,x^2-.006\,x^3+\cdots).$

Carry nothing beyond the third decimal place, and retain nothing beyond the term in x^3.

§ 12. PROB. 4.

72. $3\,a^2b:ab;\ \ -3\,ax:-x^2;\ \ mn^{-2}:-m^2n;\ \ -r^3st^{-1}:2\,r^{-2}s^2t^{-2}.$

73. $(x^2+2\,ax+b):x;\ \ (\frac{1}{2}x^2-\frac{2}{3}xy^{-2}+\frac{3}{4}y^{-4}):-3\,x^3y^{-2}.$

74. $(y^2+5\,y+6):(y+2);\ \ (15\,x^3+x^2y^{-1}+4\,y^{-3}):(3\,x+2\,y^{-1}).$

75. $(a^{m+n}-a^mb^n+a^nb^m-b^{m+n}):(a^n-b^n).$

76. $$\overline{\begin{array}{r|r|r|l} x^4+a & x^3+ab & x^2+abc & x+abcd \\ +b & +ac & +abd & \\ +c & +ad & +acd & \\ +d & +bc & +bcd & \\ & +bd & & \\ & +cd & & \end{array}} \div \overline{\begin{array}{r|l} x^2+a & x+ab. \\ +b & \end{array}}$$

§ 12. PROB. 4, NOTE 2.

77. $x:(x+a)$, $(a+x):(b+x)$, $a^2:(a+x)^2$, $a:(1+x)$, $(1+2x):(1-3x)$, $1:(1-2x+x^2)$, each to four places of the quotient, then write the complete quotients by annexing the remainders written over the divisors.

78. $(1+x^3-8y^3+6xy):(1+x-2y)$, first to ascending powers of x, second to descending powers of y.
Test by multiplication.

79. $(18xyz+27z^3-x^3+8y^3):(x-3z-2y)$.
Test severally by all the principles in [th. 5. crs. 2–7].

80. Show that if x^2+px+q be divided by $x-a$, the remainder is the value the dividend has when a is substituted for x.

81. So, if x^3+px^2+qx+r be divided by $x-a$.

82. So, if $x^4+px^3+qx^2+rx+s$ be divided by $x-a$.

§ 12. PROB. 4, NOTES 3, 4.

83–86. Divide by detached coefficients, and by synthetic division:

83. $\overline{x^3-9x^2+26x-24}:\overline{x-4}$; $\overline{2x^3-4x^2-3x+11}:\overline{x-2}$.

84. $(x^4-3x^2y^2+xy^3):(x-3y):(x+y)$.

85. $(2x^5+10x^4-7x^3-14x^2+11x-2):(x^2+5x-2):(x^3-x+1)$.

86. $(1+2x^3):(1+x+x^2)$; $(x^{2a}+y^{2b}):(x^a+y^b)$, each to four terms of the quotient; write the complete quotients.
Make two divisions, the first to ascending powers of x, the other to descending powers of x.

87. $(x^5-2x^4+3x^3-4x^2+5x-6):(x-2)$, the quotient $:(x-2)$, $\cdots$; write the last quotient and the numerical remainders in a series; use synthetic division.

88–101. Divide as shown by the signs; follow the processes given in the notes exemplified:

§ 12. PROB. 4, NOTE 5.

88. $(x^4 - y^4) : (x^2 - y^2)$; $(4x^6 - 9y^6) : (2x^3 + 3y^3)$; $(x^{2n} - y^{2n}) : (x^n \pm y^n)$.

89. $(a^{2n} - b^{2n}) : (a + b)$; $(a^{2n+1} + b^{2n+1}) : (a + b)$.

90. $(a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8) : (a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

91. $[(x^3 + a^3) \cdot (x^3 - a^3)] : [(x^2 + ax + a^2) \cdot (x^2 - ax + a^2)]$.

92. $(\overline{a+b}^2 - c^2) : (a + b - c)$; $(a^2 - \overline{b-c}^2) : (a - b + c)$.

93. $(\overline{x+y}^3 + z^3) : (x + y + z)$; $(x^3 - \overline{y-z}^3) : (x - y + z)$.

94. $(x^{mn} - 1) : (x^m - 1)$; $(x^{mn} - 1) : (x^n - 1)$.

§ 12. PROB. 4, NOTE 6.

95. $9fg : (3f^{\frac{1}{2}}g^{\frac{1}{2}} \pm 2u^{\frac{1}{2}}v^{\frac{1}{2}})^2$; $\overline{a+b}^2 : (\overline{a+b} \pm 3 \cdot \overline{x+y})^2$; by making suitable substitutions in Ex. 77. Get the quotients as far as the cubes of $u^{\frac{1}{2}}v^{\frac{1}{2}}$ and $x + y$, then write the complete quotients.

96. $$\overline{\begin{array}{r|r|r|r} a & x^3 + 2a^2 & x^2 + 2a^3 & x + a^4 \\ +b & +4ab & +6a^2b & +4a^3b \\ & +2b^2 & +6ab^2 & +6a^2b^2 \\ & & +2b^3 & +4ab^3 \\ & & & +b^4 \end{array}} \;\Bigg|\; \div \overline{\begin{array}{r|r} x^2 + a & x + a^2 \\ +b & +2ab \\ & +b^2 \end{array}}\;\Bigg|$$

§ 12. PROB. 4, NOTE 7.

97. $(x^2 + 2x - 13 + 2x^{-1} + x^{-2}) : (x + 5 + x^{-1})$.

98. $(a^3 + b^3 + c^3 - 3abc) : (a + b + c)$;
$(a^3 - b^3 + c^3 + 3abc) : (a - b + c)$.

99. $[x^3(y - z) + y^3(z - x) + z^3(x - y)]$
$: [x^2(y - z) + y^2(z - x) + z^2(x - y)]$.

§ 12. PROB. 4, NOTE 8.

100. $(1 - .2x + .04x^2 - .008x^3 + \cdots) : (1 + .1x + .01x^2 + .001x^3 + \cdots)$. Carry nothing beyond the third decimal place, nor beyond x^3.

101. $(x^3 + 11x^2 - 102x + 181) : (x - 3.213)$, with same limitation as in Ex. 100.

102. If N be any dividend; $D_1, D_2, \cdots$ any divisors; Q_1, R_1 the quotient and remainder got by dividing N by D_1; Q_2, R_2 the quotient and remainder got by dividing Q_1 by $D_2, \cdots$, show that

$$\begin{aligned} N, &= Q_1 \cdot D_1 + R_1, \\ &= (Q_2 D_2 + R_2) \cdot D_1 + R_1 = Q_2 \cdot D_2 D_1 + R_2 \cdot D_1 + R_1, \\ &= (Q_3 D_3 + R_3) \cdot D_2 D_1 + R_2 \cdot D_1 + R_1 \\ &= Q_3 D_3 D_2 D_1 + R_3 \cdot D_2 D_1 + R_2 \cdot D_1 + R_1, \text{ and so on,} \\ &= Q_n D_n D_{n-1} \cdots D_3 D_2 D_1 + R_n \cdot D_{n-1} \cdots D_2 D_1 + \cdots \\ &\qquad + R_2 \cdot D_1 + R_1; \end{aligned}$$

and if $D_1 = D_2 = D_3 = \cdots$, then that

$$\begin{aligned} N = Q_n D_1^n + R_n D_1^{n-1} + R_{n-1} D_1^{n-2} + \cdots \\ + R_3 D_1^2 + R_2 D_1 + R_1. \end{aligned}$$

103. By the method of Ex. 102 develop $x^4+8x^3+24x^2+32x+16$ to powers of $x+1$; of $x-1$; of $x+3$; of x^2+x+1; also in the form $Ax + Bx(x+1) + Cx(x+1)(x+2) + Dx(x+1)(x+2)(x+3)$, wherein A, B, C, D are free from x.

104. Express $\dfrac{3x^3-16x^2+24x-1}{(x-2)^4}$ as a sum of fractions whose numerators are free from x.

First solution: Develop the numerator to powers of $x-2$ [Ex. 102], viz., $3(x-2)^3+2(x-2)^2-4(x-2)+7$; then,

$$\begin{aligned} \frac{3x^3-16x^2+24x-1}{(x-2)^4} &= \frac{3(x-2)^3}{(x-2)^4} + \frac{2(x-2)^2}{(x-2)^4} - \frac{4(x-2)}{(x-2)^4} + \frac{7}{(x-2)^4} \\ &= \frac{3}{x-2} + \frac{2}{(x-2)^2} - \frac{4}{(x-2)^3} + \frac{7}{(x-2)^4}. \text{ Q.E.F.} \end{aligned}$$

Second solution: Divide both numerator and denominator by $x-2$ three times in succession; then,

$$\begin{aligned} \frac{3x^3-16x^2+24x-1}{(x-2)^4} &= \frac{3x^2-10x+4}{(x-2)^3} + \frac{7}{(x-2)^4} \\ &= \frac{3x-4}{(x-2)^2} - \frac{4}{(x-2)^3} + \frac{7}{(x-2)^4} \\ &= \frac{3}{x-2} + \frac{2}{(x-2)^2} - \frac{4}{(x-2)^3} + \frac{7}{(x-2)^4}. \end{aligned}$$

105. Express $\dfrac{x^3+x^2+x+1}{(x-3)^4}$ as a sum of fractions whose numerators are free from x.

106. Express $\dfrac{x^3+x^2+x+1}{(x-3)^2}$ as a sum of entire terms, and of fractions whose numerators are free from x.

107. Express $\dfrac{x^4-4x^2+1}{x(x+1)(x+2)}$ as a sum of entire terms and of fractions whose numerators are free from x, and whose denominators are x, $x(x+1)$, $x(x+1)(x+2)$; either by first developing the numerator as to $x+2$, $(x+1)(x+2)$, $x(x+1)(x+2)$, or by dividing both numerator and denominator successively by $x+2$, $x+1$, x.

108. Express $\dfrac{x^4-4x+1}{x(x-1)(x-2)}$ as a sum of entire terms and of fractions whose numerators are free from x, and whose denominators are x, $x(x-1)$, $x(x-1)(x-2)$.

§ 13. PROB. 5.

109–112. Reduce to lower terms the fractions:

109. $\dfrac{x^2+3x+2}{x^2+4x+3}$; $\dfrac{x^2-3x+2}{x^2-4x+3}$; $\dfrac{x^2-2x-15}{x^2+2x-35}$; $\dfrac{acx^2+(ad-bc)x-bd}{a^2x^2-b^2}$.

110. $\dfrac{a^2-b^2}{a^4-b^4}$; $\dfrac{a^2-b^2}{a^2\pm 2ab+b^2}$; $\dfrac{4x^2-9}{4x^2\pm 12x+9}$.

111. $\dfrac{4x^2-(3y-4z)^2}{(2x+3y)^2-16z^2}$; $\dfrac{(4x^2+3x+2)^2-(2x^2+3x+4)^2}{(3x^2+x-1)^2-(x^2-x-3)^2}$.

112. $\dfrac{m^3-n^3}{m^4-n^4}$; $\dfrac{p^4-q^4}{p^5-q^5}$; $\dfrac{r^5-s^5}{r^6-s^6}$; $\dfrac{x^{2n}-y^{2n}}{x^{3n}-y^{3n}}$; $\dfrac{x^{2n}-y^{-2n}}{x^{3n}-y^{-3n}}$.

§ 13. PROB. 6.

113. Reduce to equivalent fractions, with the common numerator a^4-b^4, the fractions:

$\dfrac{a-b}{a+b}$; $\dfrac{a+b}{a-b}$; $\dfrac{a^2-b^2}{a^2+b^2}$; $\dfrac{a^2+b^2}{a^2-b^2}$; $\dfrac{a^3+a^2b+ab^2+b^3}{a^3-a^2b+ab^2-b^3}$; $\dfrac{a^3-a^2b+ab^2-b^3}{a^3+a^2b+ab^2+b^3}$.

114. Reduce to equivalent fractions with the common denominator $\overline{\begin{array}{r|r|l} x^3+a & x^2+ab & x+abc, \\ +b & +ac & \\ +c & +bc & \end{array}}$ the fractions:

$$\frac{\begin{array}{r|l} x^2-b & x+bc \\ -c & \end{array}}{x+a}; \quad \frac{\begin{array}{r|l} x^2-a & x+ac \\ -c & \end{array}}{x+b}; \quad \frac{\begin{array}{r|l} x^2-a & x+ab \\ -b & \end{array}}{x+c};$$

$$\frac{x-c}{\begin{array}{r|l} x^2+a & x+ab \\ +b & \end{array}}; \quad \frac{x-b}{\begin{array}{r|l} x^2+a & x+ac \\ +c & \end{array}}; \quad \frac{x-a}{\begin{array}{r|l} x^2+b & x+bc \\ +c & \end{array}}.$$

§ 13. PROB. 7.

115. Reduce to a common denominator the sets of fractions:

$$\frac{1}{xy}, \frac{2}{yz}, \frac{3}{zx}; \quad \frac{x}{1-x^2}, \frac{x}{(1-x)^2}; \quad \frac{x^2}{a^2+ax}, \frac{a^2}{x^2-ax}, \frac{ax}{a^2-x^2};$$

$$\frac{a}{a-x}, \frac{3a}{a+x}, \frac{2ax}{a^2-x^2}; \quad \frac{2}{x}, \frac{3}{2x-1}, \frac{2x-3}{4x^2-1};$$

$$\frac{a^2-bc}{(a+b)(a+c)}, \frac{b^2-ca}{(b+c)(b+a)}, \frac{c^2-ab}{(c+a)(c+b)}.$$

§ 13. PROB. 8.

Add and subtract, as shown by the signs, the fractions:

116. $1+\dfrac{1}{1+x}+\dfrac{1}{1-x}; \quad 1-\dfrac{1}{1+x}-\dfrac{1}{1-x}; \quad \dfrac{1-x}{1+x}+\dfrac{1+x}{1-x}.$

117. $\dfrac{a+b}{a+x}-\dfrac{a-b}{a-x}+\dfrac{a+b}{a-x}-\dfrac{a-b}{a+x}; \quad \dfrac{1}{a+b}+\dfrac{b}{a^2-b^2}-\dfrac{a}{a^2+b^2}.$

118. $\dfrac{1}{(a-b)(a-c)}+\dfrac{1}{(b-c)(b-a)}+\dfrac{1}{(c-a)(c-b)}.$

119. $\dfrac{a^3}{(a-b)\cdot(a-c)}+\dfrac{b^3}{(b-c)\cdot(b-a)}+\dfrac{c^3}{(c-a)\cdot(c-b)}.$

120. $\dfrac{x^{3n}}{x^n-1}-\dfrac{x^{2n}}{x^n+1}-\dfrac{x^n}{x^n-1}+\dfrac{x^0}{x^n+1}.$

121. $\dfrac{y^2c^2}{b^2c^2}+\dfrac{(y^2-b^2)\cdot(z^2-b^2)}{b^2(b^2-c^2)}+\dfrac{(y^2-c^2)\cdot(z^2-c^2)}{c^2(c^2-b^2)}.$

§ 13. PROB. 9.

122–124. Multiply and divide, as shown by the signs, the following fractions. Take care to keep every fraction in its lowest terms, and to cancel where possible:

122. $\dfrac{a^2-b^2}{x-y}\cdot\dfrac{x^2-y^2}{a-b}\cdot\dfrac{c^2}{x-y}$; $\left(1+\dfrac{1}{x}\right):\left(x-\dfrac{1}{x}\right)\cdot\left(1-\dfrac{1}{x}\right)^2$.

123. $\dfrac{x^4-b^4}{x^2-2bx+b^2}:\dfrac{x^2+bx}{x-b}\cdot\dfrac{x^5-b^2x^3}{x^3+b^3}:\dfrac{x^4-2bx^3+b^2x^2}{x^2-bx+b^2}$.

124. $\dfrac{a^3-x^3}{a^3+x^3}\cdot\dfrac{a^2-x^2}{a^2+x^2}\cdot\dfrac{a-x}{a+x}\cdot\dfrac{a^2-ax+x^2}{a^2+ax+x^2}\cdot\dfrac{a^2+2ax+x^2}{a^2-2ax+x^2}$.

125. Show that $a:b:c=\overline{a:b}:c=a:\overline{b\times c}$. [th. 3 cr. 8

126. Show that $a:b:c:d=\overline{a:b:c}:d$

$$=\overline{a:b}:\overline{c\times d}=a:\overline{b\times c\times d}.$$

127. Show that $\dfrac{n}{d}\cdot\dfrac{n'}{d'}:\dfrac{n''}{d''}=\dfrac{n\cdot n':n''}{d\cdot d':d''}=\dfrac{n\cdot n'\cdot d''}{d\cdot d'\cdot n''}$

$$=\overline{n:\overline{n'':d''}}:\overline{d:\overline{d':n'}}=n:d:\overline{\overline{n':\overline{d':\overline{d'':n''}}}}.$$

128. Remove the bars and reduce $2a:\overline{3a^2\cdot\overline{4a^3:6a^4\cdot a^3:a^2}\cdot a}$.

129. Remove the brackets and reduce to lowest terms:

$$\left[\left(x+\frac{x+1}{x}\right):\left(1+\frac{a}{b}\right)\right]:\left[\tfrac{1}{2}b:\left(1+\frac{b}{a}\right):4ax:\left(x+\frac{1}{x+1}\right)\right].$$

130. Reduce the following complex fractions to simple fractions,
(*a*) by performing the operations indicated,
(*b*) by multiplying both numerator and denominator by a suitable multiplier:

$$\frac{x-\dfrac{x-y}{1+xy}}{1+\dfrac{x(x-y)}{1+xy}};\quad \frac{\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}};\quad \frac{\dfrac{m^2+mn+n^2}{m^3+n^3}}{\dfrac{m^3-n^3}{m^2-mn+n^2}};\quad \frac{\dfrac{p^6+q^6}{p^3+q^3}}{\dfrac{p^3-q^3}{p^6-q^6}};$$

$$\frac{1}{1-\dfrac{1}{1+x}};\quad \frac{1}{1-\dfrac{1}{1-x}};\quad \frac{1}{1+\dfrac{1}{1+\dfrac{1}{x}}};\quad \frac{\dfrac{m+n}{m-n}+\dfrac{m-n}{m+n}}{\dfrac{m-n}{m+n}-\dfrac{m+n}{m-n}}.$$

131–136. As an exercise on fractions, prove the theorems of proportion [ths. 6–9] and their corollaries.

131. If $\frac{a}{b}=\frac{c}{d}$, then will $ad=bc$; if $\frac{a}{b}=\frac{b}{c}$, then will $ac=b^2$.

132. If $ad=bc$, then will $\frac{a}{b}=\frac{c}{d}$; if $ac=b^2$, then will $\frac{a}{b}=\frac{b}{c}$.

133. If $\frac{a}{b}=\frac{c}{d}$, then will $\frac{b}{a}=\frac{d}{c}$, $\frac{a}{c}=\frac{b}{d}$, $\frac{a\pm b}{a}=\frac{c\pm d}{c}$,

$\frac{a\pm b}{b}=\frac{c\pm d}{d}$, $\frac{a+b}{a-b}=\frac{c+d}{c-d}$; and conversely.

134. If $\frac{a}{b}=\frac{c}{d}$, $\frac{a'}{b'}=\frac{c'}{d'}$, $\frac{a''}{b''}=\frac{c''}{d''}$, $\cdots$,

then will $\frac{aa'a''\cdots}{bb'b''\cdots}=\frac{cc'c''\cdots}{dd'd''\cdots}$; $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

135. If $\frac{a}{b}=\frac{c}{d}$ and $\frac{a'}{b'}=\frac{c'}{d'}$, then will $\frac{a}{b}:\frac{a'}{b'}=\frac{c}{d}:\frac{c'}{d'}$.

136. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots$, then will $\frac{a+c+e+\cdots}{b+d+f+\cdots}=\frac{a}{b}=\frac{c}{d}=\cdots$,

$\frac{ha+kc+le+\cdots}{hb+kd+lf+\cdots}=\frac{a}{b}$, $\frac{ha^n+kc^n+le^n+\cdots}{hb^n+kd^n+lf^n+\cdots}=\frac{a^n}{b^n}$.

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§§ 1–3.

137. State the converses of Axs. 1–7, and show that Ax. 1 is its own converse.

138. Show that of any simple operation, the first inverse of the first inverse is the original operation; and the second inverse of the second inverse of the second inverse is the original operation.

139. Show that if a simple operation be commutative, its first and second inverse are alike in kind.

140. Exemplify Exs. 138, 139 when $x\equiv$ operand, $k\equiv$ operator, $u\equiv$ result,

and $u=x+k$; $u=x-k$; $u=2x+3k$; $u=kx:(k+x)$; $u=x^k$.

Find the modulus when $u=\frac{kx+1}{x+k}$; when $u=\frac{2kx}{2+k}$.

III. MEASURES, MULTIPLES, AND FACTORS.

MANY of the properties of integers are shared by entire literal expressions, and the two are here treated together as *entire numbers*.

§ 1. DEFINITIONS.

WHEN the complete quotient of two numbers is entire, the $\left\{ \begin{matrix} \text{divisor} \\ \text{dividend} \end{matrix} \right.$ is a $\left\{ \begin{matrix} \textit{measure} \\ \textit{multiple} \end{matrix} \right.$ of the $\left\{ \begin{matrix} \text{dividend.} \\ \text{divisor.} \end{matrix} \right.$

E.g., 0, 3, 6, 9 are multiples of 3; 0, $3\frac{1}{2}$, $^-7$, $^\pm 10\frac{1}{2}$, of $^-3\frac{1}{2}$;
and 3 is a measure of 0, 3, 6, 9; $^-3\frac{1}{2}$, of 0, $3\frac{1}{2}$, $^-7$, $^\pm 10\frac{1}{2}$.

So $x-a$ is a measure of x^2-a^2, x^3-a^3, $b(x-a)$, but not of $x+a$;
and x^2-a^2, x^3-a^3, $b(x-a)$, but not $x+a$, are multiples of $x-a$.

The measures and multiples of a numeral depend upon its value; of a literal expression, upon its form; and one expression may measure another, but its value not measure the value of that other, or the reverse.

E.g., if $x=\frac{1}{2}$ and $a=\frac{1}{6}$,
then the value of $x-a$, $=\frac{1}{3}$, is not a measure of the value of x^2-a^2, $=\frac{2}{9}$; but is a measure of the value of $x+a$, $=\frac{2}{3}$.

When the complete quotient is entire as to the numerals, or as to any letter or letters, then the $\left\{ \begin{matrix} \text{divisor} \\ \text{dividend} \end{matrix} \right.$ is a $\left\{ \begin{matrix} \text{measure} \\ \text{multiple} \end{matrix} \right.$ of the $\left\{ \begin{matrix} \text{dividend} \\ \text{divisor} \end{matrix} \right.$ as to the numerals, or as to the same letter or letters.

E.g., $8(x-a)(y-b)$ is a measure of $2m(x^2-a^2)(y+b)$ as to a, m, and x,
and a multiple of it as to the numerals;
but neither measure nor multiple as to y, nor as to b.

So $\frac{1}{6}u^{-1}$ is a measure of $\frac{2}{3}v^{-2}$ as to the numerals and u,
and a multiple as to v.

NOTE. The words "multiple" and "measure," as here used, are an extension of "multiple" and "part," as used in I. § 8. When the quotient is an integer the two uses are identical, but they are not necessarily identical when the quotient is an entire number; for, though an integer is always an entire number, an entire number may or may not be an integer.

A *common* $\left\{\begin{matrix}\textit{measure}\\ \textit{multiple}\end{matrix}\right.$ of two or more numbers is a $\left\{\begin{matrix}\text{measure}\\ \text{multiple}\end{matrix}\right.$ of each of them; and the $\left\{\begin{matrix}\textit{highest common measure}\\ \textit{lowest common multiple}\end{matrix}\right.$ is that $\left\{\begin{matrix}\text{measure}\\ \text{multiple}\end{matrix}\right.$ which gives, for quotient, the smallest possible numeral, or the literal expression of lowest possible degree and with smallest possible coefficient.

E.g., 3 is the h. c. msr. of 6, 9, and 12, but not of 6 and 12, and 18 is the l. c. mlt. of 3, 6, and 9, but not of 3 and 9.

So $x-a$ is the h. c. msr. of x^2-a^2, x^3-a^3, and $b(x-a)$, and $b(x^6-a^6)$ is the l. c. mlt. of x^3+a^3, x^3-a^3, and $b(x-a)$.

NOTE. Strictly speaking, two or more numbers have two h. c. msrs. and two l. c. mlts., opposites of each other, either of which may be used.

E.g., 3 and -3, 18 and -18, $x-a$ and $a-x$, $b(x^6-a^6)$, and $b(a^6-x^6)$, in the above examples.

An entire number is $\left\{\begin{matrix}\textit{even}\\ \textit{odd}\end{matrix}\right.$ when $\left\{\begin{matrix}\text{a multiple}\\ \text{not a multiple}\end{matrix}\right.$ of 2.

E.g., $-6, 0, 2, 10ab$, are even; $\pm 1, 3, 5x^2$, are odd.

A $\left\{\begin{matrix}\textit{prime}\\ \textit{composite}\end{matrix}\right.$ number is an entire number that has $\left\{\begin{matrix}\text{no}\\ \text{some}\end{matrix}\right.$ entire measure besides $\pm$ itself and ± 1.

The *prime factors* of a composite number are the primes which, multiplied together, produce it; and *to factor* a composite number is to find all its prime factors.

E.g., $600a^4x^2-600a^2x^4 = 2^3\cdot 3\cdot 5^2\cdot a^2\cdot x^2\cdot (a+x)\cdot (a-x)$, twelve prime factors.

Entire numbers are *prime to each other* when they have no entire common measure except ± 1.

E.g., 9, 10, a^2, x^3, $7y$, x^2-y^2, are all composite, but prime to each other.

§ 2. AXIOMS.

1. Every number is both a measure and a multiple of itself, and of its opposite.

2. ± 1 is a common measure of all entire numbers, and a common multiple of their reciprocals.

3. 0 is a common multiple of all other numbers.

4. Every $\left\{\begin{matrix}\text{measure of a numeral not 0,}\\ \text{multiple, not 0, of a numeral,}\end{matrix}\right.$ is at least as $\left\{\begin{matrix}\text{small}\\ \text{large}\end{matrix}\right.$ as that numeral.

5. Every multiple, not 0, of a number contains all the letters of that number, and to at least as high a degree as the number itself.

6. All prime numbers are prime to each other; but not all numbers prime to each other are primes.

7. ± 1 is a prime, and is prime to all entire numbers.

8. An entire function of entire numbers is an entire number.

§ 3. MEASURES AND MULTIPLES.

THEOR. 1. *Any* $\left\{\begin{matrix}\textit{measure}\\ \textit{multiple}\end{matrix}\right.$ *of a number is a* $\left\{\begin{matrix}\textit{measure}\\ \textit{multiple}\end{matrix}\right.$ *of any* $\left\{\begin{matrix}\textit{multiple}\\ \textit{measure}\end{matrix}\right.$ *of that number.*

Let A be any number, and $\left\{\begin{matrix}\text{M a msr.}\\ \text{N a mlt.}\end{matrix}\right.$ of A, then is $\left\{\begin{matrix}\text{M a msr. of N.}\\ \text{N a mlt. of M.}\end{matrix}\right.$

For $\because$ the quotients $A:M$ and $N:A$ are entire, [hyp.

$\therefore$ $\overline{A:M} \times \overline{N:A}, = N:M$, is entire. Q.E.D. [ax. 8

NOTE. Th. 1 may also be stated thus:

Any $\left\{\begin{matrix}\text{measure of a measure}\\ \text{multiple of a multiple}\end{matrix}\right.$ of a number is a $\left\{\begin{matrix}\text{measure}\\ \text{multiple}\end{matrix}\right.$ of that number.

COR. 1. *Any* $\left\{\begin{matrix}\textit{measure}\\ \textit{multiple}\end{matrix}\right.$ *of a number is a* $\left\{\begin{matrix}\textit{measure}\\ \textit{multiple}\end{matrix}\right.$ *of its opposite.*

COR. 2. *An even number cannot measure an odd number;* i.e., *any* $\left\{\begin{matrix}\textit{measure}\\ \textit{multiple}\end{matrix}\right.$ *of an* $\left\{\begin{matrix}\textit{odd}\\ \textit{even}\end{matrix}\right.$ *number is* $\left\{\begin{matrix}\textit{odd.}\\ \textit{even.}\end{matrix}\right.$

THEOR. 2. *A common measure of two or more numbers is a measure of their sum.*

Let A, B, $\cdots$ be any numbers and M a common measure of them, then is M a measure of $A + B + \cdots$.

For $\because$ $A = a \cdot M$, $B = b \cdot M$, $\cdots$, wherein $a, b, \cdots$ are entire, [hyp.

$\therefore$ $A + B + \cdots = a \cdot M + b \cdot M + \cdots$ [II. ax. 2

$= (a + b + \cdots) \cdot M$; [II. th. 4

and $\because$ $a + b + \cdots$ is entire, [ax. 8

$\therefore (A + B + \cdots) : M, \; = a + b + \cdots$, [I. § 9 df.

is an entire number. Q. E. D.

COR. 1. *A common measure of two numbers is a measure of their difference.*

COR. 2. *If a number measure the sum of two or more numbers, and measure all but one of them, it measures that one also.*

COR. 3. *A common measure of two or more numbers is a measure of the sum of any multiples of them.*

COR. 4. *A common measure of two numbers is a measure of the difference of any multiples of them.*

THEOR. 3. *If a simple monomial measure a polynomial in its simplest form, the monomial is a common measure of all its terms.*

Let $A + B + \cdots$ be a polynomial whose terms are simple and unlike, and let M, a monomial, be a measure of it; then is M a common measure of the separate terms A, B, $\cdots$.

For $\because$ A, B, $\cdots$ are unlike, [hyp.

$\therefore$ the quotients $A : M$, $B : M$, $\cdots$ are unlike,

and their sum cannot be reduced;

$\therefore$ if either of them were fractional,

then their sum, the quotient $(A + B + \cdots) : M$, would be fractional;

but $\because (A + B + \cdots) : M$ is entire and not fractional, [hyp.

$\therefore$ $A : M$, $B : M$, $\cdots$ are not fractional, but all entire. Q. E. D.

NOTE. Ths. 1, 2, 3, and their corollaries, may be extended by writing $\left\{\begin{array}{l}\text{“measure}\\ \text{“multiple}\end{array}\right.$ as to any letter or letters,” and $\left\{\begin{array}{l}\text{“measure}\\ \text{“multiple}\end{array}\right.$ as to the same letter or letters,” instead of simply $\left\{\begin{array}{l}\text{“measure.”}\\ \text{“multiple.”}\end{array}\right.$

§ 4. PRIME AND COMPOSITE NUMBERS.—FACTORS.

THEOR. 4. *If a prime number measure the product of two or more entire numbers, it measures at least one of them.*

Let A, B, ⋯ be any entire numbers, and let P, a prime, measure their product; then will P measure either A, or B, or some other one of them.

(*a*) A *and* B *two numerals,* P *a numeral.*

For, if not, divide A and B by P, and let Q, Q′ ≡ the quotients, and R, R′ ≡ the remainders, all integers;

then ∵ $A = PQ + R$ and $B = PQ' + R'$, [I. § 9 df.

∴ $AB = P^2QQ' + PQR' + PQ'R + RR'$, [II. ax. 4

i.e., $AB =$ a multiple of P, $+ RR'$, [th. 2

∴ P, a measure of AB, also measures RR′. [th. 2 cr. 2

Divide P by R, and let Q_1, R_1 ≡ quotient and remainder, both integers;

divide P by R_1, and let Q_2, R_2 ≡ quotient and remainder, both integers, and so on;

then ∵ R, R_1, R_2, ⋯ are all integers, and successively smaller and smaller,

∴ one of them, say R_k, is 0;

and ∵ P, a prime, when divided by R_{k-1}, the next preceding remainder, gives R_k, = 0, for remainder,

∴ $R_{k-1} = 1$.

But ∵ $R_1 = P - Q_1R$, $R_2 = P - Q_2R_1$, ⋯, [above

∴ $R_1R' = PR' - Q_1RR'$, $R_2R' = PR' - Q_2R_1R'$, ⋯; [II. ax. 4

and ∵ P measures RR′, [above

∴ P measures R_1R', $= PR' - Q_1RR'$, [th. 2 cr. 4

∴ P measures R_2R', $= PR' - Q_2R_1R'$, and so on;

∴ P measures $R_{k-1}R'$, $= R'$; [above

i.e., P measures a numeral $<$ itself, which is absurd. [ax. 4

∴ the supposition that P measures neither A nor B fails,

and it is only left that P measures one of them. Q. E. D.

(*b*) A *and* B *two numerals,* P *a literal expression.*

This case cannot occur, since the numerical product A · B cannot be measured by an entire literal expression. [ax. 5

(*c*) A, *or* B, *or both of them, literal expressions*, P *a numeral.*

For, let x, y, $\cdots$ be the letters involved in A, or in B, so that A, or B, or both of them, are functions of x, y, $\cdots$;

then if P does not measure A nor B, P does not measure all the terms of A, nor of B. [th. 2

Of those terms of $\left\{ \begin{matrix} \text{A} \\ \text{B} \end{matrix} \right.$ which are not measured by P, let $\left\{ \begin{matrix} a \\ b \end{matrix} \right.$ be the numerical coefficient of that one whose degree as to x is highest,

or if there be two or more such terms, then of that one of them whose degree as to y also is highest, and so on;

then $\because$ $a \cdot b$ is the coefficient of a term, T, of A · B, which has a higher degree as to its own letters x, y, $\cdots$ than the degree of any other term as to those letters, and is like no other term;

$\therefore$ T combines with no other term, and remains unchanged when the polynomial is in its simplest form.

But $\because$ P measures neither a nor b, [hyp.

$\therefore$ P does not measure their product $a \cdot b$, [(*a*)

$\therefore$ not the term T of A · B,

$\therefore$ not A · B, [th. 3

which is contrary to the hypothesis of the theorem;

$\therefore$ the supposition that P measures neither A nor B fails,

and it is only left that P measures one of them. Q. E. D.

(*d*) A, *or* B, *or both of them, literal*, P *literal.*

For, if not, let x be a letter found in P;

then x is found also in the product A · B, [ax. 5

and $\therefore$ in either A, or B, or both of them.

Arrange A, B, and P, by descending powers of x, divide A and B by P, and let Q, Q$'$ $\equiv$ the quotients, and R, R$'$ $\equiv$ the remainders, all entire as to x;

then $\because$ $\text{A} = \text{PQ} + \text{R}$ and $\text{B} = \text{PQ}' + \text{R}'$, [I. § 9, df.

$\therefore$ $\text{AB} = \text{P}^2\text{QQ}' + \text{PQR}' + \text{PQ}'\text{R} + \text{RR}'$, [II. ax. 4

i.e., $\text{AB} =$ a multiple of P as to x, $+ \text{RR}'$, [th. 2, th. 3 nt.

$\therefore$ P, a measure of AB, also measures RR$'$ as to x. [th. 2 cr. 2

Divide P by R, and let Q_1, $R_1 \equiv$ quotient and remainder, both entire as to x;

divide P by R_1, and let Q_2, $R_2 \equiv$ quotient and remainder, both entire as to x, and so on;

then $\because$ P, being prime, has not x in every term, [th. 2

and $\because$ R, R_1, R_2, $\cdots$ are entire as to x, and of successively lower and lower degree,

$\therefore$ one of them, say R_{k-1}, is free from x.

But $\because$ $R_1 = P - Q_1R$, $R_2 = P - Q_2R_1$, $\cdots$, [above

$\therefore$ $R_1R' = PR' - Q_1RR'$, $R_2R' = PR' - Q_2R_1R'$, $\cdots$, [II. ax. 4

and $\because$ P measures RR' as to x, [above

$\therefore$ P measures R_1R', $= PR' - Q_1RR'$, as to x,

$\therefore$ P measures R_2R', $= PR' - Q_2R_1R'$, as to x, and so on;

$\therefore$ P measures $R_{k-1}R'$, as to x;

i.e., P measures an expression of lower degree as to x than P itself, which is absurd; [ax. 5

$\therefore$ the supposition that P measures neither A nor B fails,

and it is only left that P measures one of them. Q. E. D.

(*e*) *Three or more factors*, A, B, C, $\cdots$ L.

For, if P measures the product $A \cdot B \cdot C \cdots L$,

then P measures either A or the product $B \cdot C \cdots L$, [above

if P measures the product $B \cdot C \cdots L$,

then P measures either B or the product $C \cdots L$, [above

and so on;

$\therefore$ P measures either A, or B, or C, or $\cdots$, or L;

i.e., P measures one of them. Q. E. D.

COR. 1. *If a prime measure the product of two numbers, and be prime to one of them, it measures the other.*

COR. 2. *If there be two or more entire numbers, and if* P, *a prime, measure neither of them, it does not measure their product; and if not their product, then neither of them. In particular, a prime cannot measure a product of other primes.*

COR. 3. *A product of entire factors* $\left\{ \begin{array}{l} \textit{some even} \\ \textit{all odd} \end{array} \right.$ *is* $\left\{ \begin{array}{l} \textit{even.} \\ \textit{odd.} \end{array} \right.$

COR. 4. *If a prime measure a positive integral power of an entire number, it measures that number; and if the number, then the power.*

THEOR. 5. *A composite number can be resolved into one, and but one, set of prime factors.*

(*a*) *Into one set.*

For, let N be any composite number, M an entire measure of N, and Q the quotient,

then $\because$ $N = M \cdot Q$, [I. § 9 df.

$\therefore$ if M and Q be primes, N is resolved as required.

But if either, or both of them, be composite,

then they also may be resolved, and so on.

Finally, when all the prime factors, A, B, ⋯, are found, if A occur a times, B b times, ⋯,

then $N = A^a \cdot B^b \cdots$. Q. E. D.

(*b*) *Into but one set.*

For, if possible, let $N = A^a \cdot B^b \cdots$, also $= G^g \cdot H^h \cdots$, wherein A, B, ⋯ are unequal primes, and so are G, H, ⋯, and G^g, H^h, ⋯ are not wholly the same as A^a, B^b, ⋯;

then $\because$ some prime P occurs p times in one set, and not p times in the other set,

$\therefore$ of the equal quotients $A^a \cdot B^b \cdots : P^p$ and $G^g \cdot H^h \cdots : P^p$ one is entire and the other fractional, which is absurd;

$\therefore$ the supposition fails that N can be resolved into two different sets of prime factors. Q. E. D.

ENTIRE NUMBERS PRIME TO EACH OTHER.

COR. 1. *If two entire numbers have no common prime factor, they are prime to each other.*

COR. 2. *If there be two sets of entire numbers, such that each number of the first set is prime to each number of the second set, then is the product of the first set prime to the product of the second set; and conversely.*

COR. 3. *If two entire numbers be prime to each other, so are any positive integral powers of them; and conversely.*

COR. 4. *If there be two entire numbers prime to each other, any common multiple of them is a multiple of their product.*

For, let the products $A^a \cdot B^b \cdots$, $G^g \cdot H^h \cdots$, be any numbers prime to each other, and let M be a common multiple of them;

then ∵ among the prime factors of M, A occurs a times, B b times, ⋯, G g times, H h times, ⋯,

and ∵ the primes A, B, ⋯, G, H, ⋯ are all different, [hyp.

$\therefore\ \text{M} = \text{Q} \times \text{A}^a \cdot \text{B}^b \cdots \times \text{G}^g \cdot \text{H}^h \cdots,$

wherein Q is some entire number, perhaps 1. Q. E. D. [II. th. 3

So if there be three or more entire numbers prime to each other.

COMMON MEASURES AND MULTIPLES.

COR. 5. *The* $\left\{\begin{array}{l}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{array}\right.$ *of two or more entire numbers is the product of their different prime factors, each factor having the* $\left\{\begin{array}{l}\textit{least}\\ \textit{greatest}\end{array}\right.$ *exponent which it has in any of the numbers.*

COR. 6. *If there be two or more sets of entire numbers, the* $\left\{\begin{array}{l}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{array}\right.$ *of the* $\left\{\begin{array}{l}\textit{h. c. msrs.}\\ \textit{l. c. mlts.}\end{array}\right.$ *of the separate sets is the* $\left\{\begin{array}{l}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{array}\right.$ *of all the given numbers. In particular, the* $\left\{\begin{array}{l}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{array}\right.$ *of three or more numbers is the* $\left\{\begin{array}{l}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{array}\right.$ *of any one of them and the* $\left\{\begin{array}{l}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{array}\right.$ *of the others.*

COR. 7. *If each of two or more entire numbers be multiplied (or divided) by any same number, their* $\left\{\begin{array}{l}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{array}\right.$ *is multiplied (or divided) by that number.*

COR. 8. *The h. c. msr. of two entire numbers is not changed when either of them is* $\left\{\begin{array}{l}\textit{multiplied}\\ \textit{divided}\end{array}\right.$ *by an entire* $\left\{\begin{array}{l}\textit{number}\\ \textit{measure}\end{array}\right.$ *prime to the other.*

For ∵ the prime $\left\{\begin{array}{l}\text{number multiplied into}\\ \text{measure stricken out of}\end{array}\right.$ the one is not a factor of the other,

∴ it $\left\{\begin{array}{l}\text{is not}\\ \text{could not be}\end{array}\right.$ a factor of their h. c. msr. Q. E. D.

COR. 9. *The product of two entire numbers equals the product of their h. c. msr. and l. c. mlt.*

For, let N, $= \text{A}^a \cdot \text{B}^b \cdots$, and N′, $= \text{A}^{a'} \cdot \text{B}^{b'} \cdots$, be any two numbers, wherein A, B, ⋯, are primes, and the exponents $a, b, \cdots, a', b', \cdots$, are integers,

then $\because$ of the exponents a, a', the $\begin{cases}\text{less}\\\text{greater}\end{cases}$ is the exponent of A in the $\begin{cases}\text{h. c. msr.}\\\text{l. c. mlt.}\end{cases}$

$\therefore$ $a+a'$ is the exponent of the factor A in the product of the h. c. msr. by the l. c. mlt. [II. th. 3 cr. 10

So $b+b'$ is the exponent of B in that product, and so on;

$\therefore$ the product h. c. msr. $\times$ l. c. mlt. $= A^{a+a'} \cdot B^{b+b'} \cdots$.

But $\because$ $N \cdot N' = A^{a+a'} \cdot B^{b+b'} \cdots$, [II. th. 3

$\therefore$ $N \cdot N' =$ h. c. msr. $\times$ l. c. mlt. Q. E. D. [II. ax. 1

COR. 10. *Every common* $\begin{cases}\textit{measure}\\\textit{multiple}\end{cases}$ *of two or more numbers is a* $\begin{cases}\textit{measure}\\\textit{multiple}\end{cases}$ *of their* $\begin{cases}\textit{h. c. msr.}\\\textit{l. c. mlt.}\end{cases}$

APPLICATION TO FRACTIONS.

COR. 11. *If the terms of a simple fraction be prime to each other, the fraction cannot be reduced to an equivalent simple fraction in lower terms.*

For, let $\frac{A^a \cdot B^b \cdots}{G^g \cdot H^h \cdots}$ be a fraction, wherein A, B, ⋯, G, H, ⋯ are all different primes, and let $\frac{P}{Q}$ be any equivalent simple fraction;

then $\because$ $\frac{A^a \cdot B^b \cdots}{G^g \cdot H^h \cdots} = \frac{P}{Q}$, [hyp.

$\therefore$ $A^a \cdot B^b \cdots \times Q = G^g \cdot H^h \cdots \times P$, [II. ax. 4

whose two members, being the same number, can be factored in only one way; [th.

$\therefore$ among the factors of $\begin{cases}P\\Q\end{cases}$ are $\begin{cases}A^a, B^b, \cdots,\\G^g, H^h, \cdots,\end{cases}$

$\therefore$ $\begin{cases}P\\Q\end{cases}$ is a multiple of $\begin{cases}A^a \cdot B^b \cdots,\\G^g \cdot H^h \cdots,\end{cases}$

and $\frac{P}{Q}$ is not in lower terms than $\frac{A^a \cdot B^b \cdots}{G^g \cdot H^h \cdots}$. Q. E. D.

COR. 12. *If a fraction be in its lowest terms, so is every integral power of it; and conversely.*

COR. 13. *A fraction can be resolved into but one set of factors and divisors,* A^a, B^b, ⋯, *wherein* A, B, ⋯ *are primes, all different, and* a, b, ⋯ *are integers, some of them negative.*

NOTE. By aid of Cor. 13, Cors. 5–10 are extended and applied to fractions as well as entire numbers.

COR. 14. *The* $\left\{\begin{matrix}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{matrix}\right.$ *of two fractions is a fraction whose numerator is the* $\left\{\begin{matrix}\textit{h. c. msr.}\\ \textit{l. c. mlt.}\end{matrix}\right.$ *of their numerators, and whose denominator is the* $\left\{\begin{matrix}\textit{l. c. mlt.}\\ \textit{h. c. msr.}\end{matrix}\right.$ *of their denominators.*

For, let $\frac{N}{D}, = A^a \cdot B^b \cdots, \frac{N'}{D'}, = A^{a'} \cdot B^{b'} \cdots$, be any two fractions, wherein A, B, ⋯ are primes, and the exponents $a, b, \cdots, a', b', \cdots$, are integers, some of them negative or zero;

and let $\left\{\begin{matrix} a_1, b_1, \cdots \\ a_2, b_2, \cdots \end{matrix}\right.$ be the $\left\{\begin{matrix}\text{least}\\ \text{greatest}\end{matrix}\right.$ exponents in the pairs of exponents $a, a', \; b, b', \; \cdots$;

then ∵ the $\left\{\begin{matrix}\text{h. c. msr.}\\ \text{l. c. mlt.}\end{matrix}\right.$ has every measure that is in $\left\{\begin{matrix}\text{both}\\ \text{either}\end{matrix}\right.$ of the fractions, and has no others, [§ 1 dfs.

∴ the $\left\{\begin{matrix}\text{h. c. msr.}\\ \text{l. c. mlt.}\end{matrix}\right.$ of $\frac{N}{D}, \frac{N'}{D'}$, is the product $\left\{\begin{matrix} A^{a_1} \cdot B^{b_1} \cdots, \\ A^{a_2} \cdot B^{b_2} \cdots, \end{matrix}\right.$

wherein those factors which have negative exponents make up the denominator of the $\left\{\begin{matrix}\text{h. c. msr.}\\ \text{l. c. mlt.}\end{matrix}\right.$ sought. Q. E. D.

So for three or more fractions.

PRIME AND COMPOSITE MEASURES.

COR. 15. *The entire number* $A^a \cdot B^b \cdots$ *has* $(a+1) \cdot (b+1) \cdots$ *different entire measures, prime and composite (and their opposites), whose sum is* $[(A^{a+1}-1):(A-1)] \cdot [(B^{b+1}-1):(B-1)] \cdots$.

For ∵ A^a has $(a+1)$ measures, $A^a, A^{a-1}, A^{a-2}, \cdots, A^2, A^1, 1$,

and ∵ B^b has $(b+1)$ measures, and so on,

and ∵ the several products got by multiplying the $(a+1)$ measures in turn by the $(b+1)$ measures, and so on, are all different one from another,

and ∵ there are $(a+1) \cdot (b+1) \cdots$ of these products, all told,

∴ there are $(a+1) \cdot (b+1) \cdots$ different measures. Q. E. D.

And ∵ the sum of the measures is the sum of all the different products of the measures $A^a, A^{a-1}, \cdots, A^1, 1$, by the measures $B^b, B^{b-1}, \cdots, B^1, 1$, by ⋯,

∴ the sum $= (A^a + \cdots + 1) \cdot (B^b + \cdots + 1) \cdots$ [II. th. 5

$= [(A^{a+1}-1):(A-1)] \cdot [(B^{b+1}-1):(B-1)] \cdots$. [II. 6

§ 5. PROCESS OF FINDING THE HIGHEST COMMON MEASURE.

PROB. 1. TO FIND THE HIGHEST COMMON MEASURE OF TWO OR MORE NUMBERS.

(*a*) *The prime factors of all the numbers known:*

Multiply together all the different prime factors, each with the least exponent it has in any one of the numbers. [th. 5 cr. 5, cr. 13 nt.

E.g., of $9a^3b^5c$, $3ab^2cd$, and $15ab^2c^6-12ab^4$, the common prime factors are 3, a, b^2; the h. c. msr. is $3\cdot a\cdot b^2$.

So of $\frac{1}{2}xy^{-1}$, $\frac{3}{4}x^2y$, $2x^3y^{-2}(x+y)$ the h. c. msr. is $\frac{1}{4}xy^{-2}$.

(*b*) *The prime factors not known; two entire numbers:*

Divide the higher number (the larger if a numeral, and that of higher degree if literal) *by the lower; the divisor by the remainder, if any; that divisor by the second remainder, and so on, till nothing remains.*

At pleasure, $\left\{\begin{array}{l}\textit{suppress from any divisor,}\\ \textit{introduce into any dividend,}\end{array}\right.$ *any entire factor that is prime to the* $\left\{\begin{array}{l}\textit{dividend}\\ \textit{divisor}\end{array}\right.$ *corresponding.*

At pleasure, suppress from any divisor and the corresponding dividend, any common measure of them; but reserve it as a factor of the final result.

The last divisor, as above, multiplied by the reserved factors, if any, is the h. c. msr. sought.

Let A and B be any two numbers, A the higher, Q the quotient of A by B; $R_1, R_2, R_3, \cdots, R_{n-1}, R_n$, the successive remainders, whereof R_n is a measure of R_{n-1}; then is R_n the h. c. msr. sought.

1. *If no factors be introduced or suppressed.*

For $\because R_1 = A - QB$, [I. § 9 df.

$\therefore$ whatever common measures A and B have, the same measures has R_1; [th. 2 cr. 4

but $\because A = R_1 + QB$,

$\therefore$ whatever common measures B and R_1 have, the same measures has A, [th. 2 cr. 3

$\therefore$ whatever common measures B and R_1 have, the same common measures, and no others, have A and B:

so whatever common measures R_1 and R_2 have, the same and no others have B and R_1,

$\therefore$ the same and no others have A and B, and so on;

so whatever common measures R_{n-1} and R_n have, the same and no others have R_{n-2} and R_{n-1}, the same and no others have R_{n-3} and R_{n-2}, and so on,

$\therefore$ the same and no others have A and B;

but $\because$ R_n is the h. c. msr. of R_{n-1} and R_n, [hyp.

$\therefore$ R_n is the h. c. msr. of A and B. Q. E. D.

2. *If factors not common be introduced or suppressed.*

For $\because$ the h. c. msr. of the given polynomials is that of any two successive remainders of the series, [1

and $\because$ the h. c. msr. of these remainders is not changed when either of them is modified by the introduction or suppression of a factor prime to the other; [th. 5 cr. 8

$\therefore$ the h. c. msr. of these two modified remainders is the h. c. msr. sought.

So for any two remainders subsequent thereto.

So for the modified R_{n-1} and R_n. Q. E. D.

3. *If a common factor be suppressed and reserved.*

For $\because$ the h. c. msr. of the given polynomials is that of any two successive remainders of the series, [1

and $\because$ when both of these remainders are modified by the suppression of a factor common to them, their h. c. msr. is divided by the same factor; [th. 5 cr. 7

$\therefore$ the product of the h. c. msr. of these two modified remainders by the suppressed factor is the h. c. msr. sought.

So for any two remainders subsequent thereto.

So for the modified R_{n-1} and R_n. Q. E. D.

E.g., to find the h. c. msr. of x^2+x-12 and $x^2-10x+21$.

$$\begin{array}{r|r|l}
x^2+x-12 & x^2-10x+21 & 1 \\
 & x^2+x-12 & \\
\hline
 & -11x+33 & (-11 \\
x^2-3x & x-3 & x+4 \\
\hline
4x-12 & &
\end{array}
\qquad or \qquad
\begin{array}{rrr|rrr}
1 & 1 & ^-12 & 1 & ^-10 & 21 \\
 & & & 1 & 1 & ^-12 \\
\hline
 & & ^-11) & & ^-11 & 33 \\
1 & ^-3 & & & 1 & ^-3 \\
\hline
4 & ^-12 & & & &
\end{array}$$

and $x-3$ is the h. c. msr. sought. Q. E. F.

So, to find the h. c. msr. of $4ax^2 + 4ax - 48a$ and $4ax^2 - 40ax + 84a$:

$\because$ $4a$ is a common factor,

and $\because$ of the remaining factors, $x^2 + x - 12$ and $x^2 - 10x + 21$, the h. c. msr. is $x - 3$, [above

$\therefore$ $4a(x-3)$, $= 4ax - 12a$, is the h. c. msr. sought. Q. E. F.

(c) *The prime factors not known; three or more entire numbers:*

Find the h. c. msr. of any two of them (preferably the two lowest), *then the h. c. msr. of this measure and the next number, and so on till all are used; the h. c. msr. last found is the h. c. msr. sought.* [th. 5 cr. 6

E.g., to find the h. c. msr. of $x^2 + x - 12$, $x^2 - 10x + 21$, and $x^3 - 6x^2 - 19x + 84$:

$\because$ of $x^2 + x - 12$ and $x^2 - 10x + 21$ the h. c. msr. is $x - 3$,

and $\because$ $x - 3$ measures $x^3 - 6x^2 - 19x + 84$,

$\therefore$ $x - 3$ is the h. c. msr. sought. Q. E. F.

(d) *Some or all of the numbers fractions:*

Divide the h. c. msr. of the entire numbers and the numerators by the l. c. mlt. of the denominators. [th. 5 cr. 14, pr. 2

E.g., to find the h. c. msr. of $\dfrac{x^2 + x - 12}{x - 5}$ and $\dfrac{x^2 - 10x + 21}{x + 5}$:

$\because$ the h. c. msr. of the numerators is $x - 3$, [above

and $\because$ the l. c. mlt. of the denominators is $x^2 - 25$, [inspection

$\therefore$ $\dfrac{x - 3}{x^2 - 25}$ is the h. c. msr. sought. Q. E. F.

NOTE 1. In the process of case (*b*) each of the remainders R_1, R_2, $\cdots$ is the sum of a multiple of the first number and a multiple of the second number.

NOTE 2. The arrangement of terms may be as to the ascending powers of some letter, or as to the descending powers, at pleasure.

E.g., $2x^3 + 11x^2 + 20x + 21$ and $x^3 - x - 6$,

or $21 + 20x + 11x^2 + 2x^3$ and $6 + x - x^3$.

That arrangement is commonly best which makes the trial divisor smallest; and at any step of the work the highest or lowest term of the divisor may be used as trial divisor at pleasure.

The work is often shortened by using detached coefficients, and sometimes by synthetic division. It is also shortened by arrangement in columns and by not writing down quotients and products, but only remainders.

E.g., to find the h. c. msr. of $2x^3+x^2-4x-3$ and $2x^2-5x+3$:

$$\begin{array}{rrrrr|rrr}
 & 2 & 1 & -4 & -3 & 2 & -5 & 3 \\
 & 2 & -5 & 3 & & & & \\
 & & 6 & -7 & & & & \\
 & & 6 & -15 & 9 & & & \\
4) & & & 8 & -12 & & & \\
 & & & 2 & -3 & 2 & -3 & \\
 & & & & & & -2 & 3
\end{array}
\quad or \quad
\begin{array}{rrrrr|rrr}
 & 2 & 1 & -4 & -3 & 2 & -5 & 3 \\
 & & 6 & -7 & & & & \\
4) & & & 8 & -12 & & & \\
 & & & 2 & -3 & & -2 & \\
 & & & & & & & 0
\end{array}$$

and $2x-3$ is the h. c. msr. sought. Q. E. F.

So the h. c. msr. of x^3+3x^2+5x+3 and x^3+6x^2+9x+4 is $x+1$:

$$\begin{array}{r|l|r}
x^3+3x^2+5x+3 & 1 & x^3+6x^2+9x+4 \\
 & & x^3+3x^2+5x+3 \\
9x^3+12x+3 & 3-7x & 3x^2+4x+1 \\
x^3-6x^2-7x & & \\
-21x^3-28x^2-7x & & \\
22x^3+22x^2 \quad (22x^2 & & \\
x+1 & 3x+1 & 3x^2+3x \\
 & & x+1
\end{array}$$

or

$$\begin{array}{rrrr|rrrr}
1 & 3 & 5 & 3 & 1 & 6 & 9 & 4 \\
1 & -6 & -7 & & & 3 & 4 & 1 \\
22 & 22 & & & & & & \\
 & & 1 & 1 & & & 1 & 1.
\end{array}$$

Q. E. F.

So by synthetic division, to find the h. c. msr. of $a^4+3a^3b+5a^2b^2+5ab^3+2b^4$ and $2a^3+5a^2b+4ab^2+b^3$:

$$\begin{array}{rrrrrr|rrrr}
 & 1 & 3 & 5 & 5 & 2 & 2 & 5 & 4 & 1 \\
 & & -4 & -10 & -8 & -2 & & & & \\
 & 6 & 15 & 12 & 3 & & & & & \\
7) & 7 & 14 & 7 & & & -2 & -4 & -2 & \\
 & 1 & 2 & 1 & & & & -1 & -2 & -1
\end{array}$$

and $a^2+2ab+b^2$ is the h. c. msr. sought. Q. E. F.

So to find the h. c. msr. of the numerals 679, 301:

$$\begin{array}{r|c|r}
679 & & \\
\underline{602} & 2 & 301 \\
77 & 3 & \underline{231} \\
\underline{70} & 1 & 70 \\
7 & 10 & \underline{70}
\end{array}
\qquad \textit{or} \qquad
\begin{array}{r|c}
679 & \\
301 & 2 \\
77 & 3 \\
70 & 1 \\
7 & 10
\end{array}$$

and the h. c. msr. sought is 7. Q. E. F.

NOTE 3. If either of the two numbers be a product of known factors, or if both of them be such products, the work in (*b*) is shortened as follows:

Let A and B be any two numbers,

and let $A \equiv A_1 \cdot A_2 \cdots A_m$ and $B \equiv B_1 \cdot B_2 \cdots B_n$,

wherein $A_1, A_2, \cdots$ are prime to each other, and so are $B_1, B_2, \cdots$,

but $A_1, A_2, \cdots B_1, B_2, \cdots$ are not necessarily primes or powers of primes;

then $\because$ every factor of the h. c. msr. of A and B which is a prime or a power of a prime, can be a factor of but one term of the series $A_1, A_2, \cdots$, and so of the series $B_1, B_2, \cdots$,

$\therefore$ the h. c. msr. of A and B is the product of the $m \cdot n$ h. c. msrs. of the pairs of numbers $A_1, B_1 \cdots$, formed by combining each of the m numbers in the series $A_1, A_2, \cdots A_m$ with each of the n numbers in the series $B_1, B_2, \cdots B_n$,

and these measures for the most part are detected by simple inspection.

E.g., if $A = A_1 \cdot A_2 \cdot A_3 \equiv (a^2 - b^2) \cdot (x^2 - y^2) \cdot (a^2 b^2 - x^2 y^2)$,

and $B = B_1 \cdot B_2 \cdot B_3 \equiv (a^3 + b^3) \cdot (x^3 + y^3) \cdot (ab + xy)$,

then $\because$ A_1 and B_1 contain only a and b, A_2 and B_2 only x and y,

and A_3 and B_3 only factors with all four letters a, b, x, and y,

$\therefore$ A_1, A_2, and A_3 are prime to each other, and so are B_1, B_2, and B_3. [ax. 5

And $\because$ A_1 is also, for the same reason, prime to B_2 and B_3, A_2 to B_1 and B_3, and A_3 to B_1 and B_2,

$\therefore$ h. c. msr. A, B = h. c. msr. (A_1, B_1) $\times$ h. c. msr. (A_2, B_2) $\times$ h. c. msr. (A_3, B_3)

$= (a + b) \times (x + y) \times (ab + xy)$. Q. E. F.

§ 6. PROCESS OF FINDING THE LOWEST COMMON MULTIPLE.

PROB. 2. TO FIND THE LOWEST COMMON MULTIPLE OF TWO OR MORE NUMBERS.

(a) *The prime factors of all the numbers known:*

Multiply together all the different prime factors, each with the greatest exponent it has in any one of the numbers. [th. 5 cr. 5, cr. 13 nt.

E.g., of $9ab^2c^{-3}$, $12a^2b^3d^4$, and $15a^4+21a^2bd$, the different prime factors, to their highest powers, are

$2^2, 3^2, a^2, b^3, c^0, d^4, 5a^2+7bd,$

$\therefore$ the l. c. mlt. is $36\cdot a^2\cdot b^3\cdot d^4\cdot(5a^2+7bd)$,

$=180a^4b^3d^4+252a^2b^4d^5$. Q. E. F.

So to find the l. c. mlt. of a^2-b^2, a^3-b^3, and a^4-b^4:

$\because$ $a^2-b^2=(a-b)\cdot(a+b)$,

$a^3-b^3=(a-b)\cdot(a^2+ab+b^2)$,

and $a^4-b^4=(a-b)\cdot(a+b)\cdot(a^2+b^2)$;

$\therefore$ the l. c. mlt. sought is

$(a-b)\cdot(a+b)\cdot(a^2+ab+b^2)\cdot(a^2+b^2)$. Q. E. F.

(b) *The prime factors not known; two entire numbers:*

Divide the product of the two numbers by their h. c. msr.; the quotient is the l. c. mlt. sought. [th. 5 cr. 9

Or, *divide either number by their h. c. msr. and multiply the quotient by the other number.*

E.g., to find the l. c. mlt. of x^2+x-12 and $x^2-10x+21$:

$\because$ their h. c. msr. is $x-3$, [pr. 1 (b) ex.

$\therefore$ $(x^2+x-12)\cdot(x^2-10x+21):(x-3)$,

$=(x^2+x-12)\cdot(x-7)$, $=x^3-6x^2-19x+84$,

is the l. c. mlt. sought. Q. E. F.

(c) *The prime factors not known; three or more entire numbers:*

Find the l. c. mlt. of any two of the numbers (preferably the two highest); *then the l. c. mlt. of this multiple and the next number, and so on, till all the numbers are used; the l. c. mlt. last found is the l. c. mlt. sought.* [th. 5 cr. 6

E.g., to find the l. c. mlt. of 289, 323, 361:

The l. c. mlt. of 289 and 323 is 5491, [(a)

and the l. c. mlt. of 5491 and 361 is 104 329. Q. E. F. [(a)

NOTE. The solutions of Pr. 1 (a) and Pr. 2 (a) extend to the h. c. msr. and l. c. mlt. of any numbers that are resolved into factors prime to one another, whether into prime factors or not.

E.g., $\because$ $3a^2-10a+3$, $6b^2+7b-20$, m^3+n^3 are all prime to one another, [inspection

$\therefore$ of $(3a^2-10a+3)^3\cdot(6b^2+7b-20)^2\cdot(m^3+n^3)$
and $(3a^2-10a+3)^2\cdot(6b^2+7b-20)^3\cdot(m^3+n^3)^{-1}$
the h. c. msr. is

$$(3a^2-10a+3)^2\cdot(6b^2+7b-20)^2\cdot(m^3+n^3)^{-1},$$

and the l. c. mlt. is

$$(3a^2-10a+3)^3\cdot(6b^2+7b-20)^3\cdot(m^3+n^3).$$

§ 7. PROCESS OF FACTORING.

PROB. 3. TO FACTOR AN ENTIRE NUMBER.

IN GENERAL.

Take out all monomial factors by inspection; by inspection also, or by trial, find an entire measure of the remaining factors; then of this measure, and of its co-factor; and so on, till no composite factor remains. Write the prime factors in order, and mark each one of them with that exponent which shows how many times it has been used.

IN PARTICULAR.

(a) *The number an integer:*

Divide the number, and the successive quotients in order, by the primes 2, 3, 5, ⋯, *using each divisor as many times as it measures the successive dividends. The successful divisors, and the last undivided dividend, are the prime factors sought.*

NOTE. No divisor larger than the square root of the dividend need be tried.

For $\because$ dividend $=$ divisor $\times$ quotient, [I. § 9 df.

$\therefore$ if divisor $\gt \sqrt{}$dividend, then quotient $\lt \sqrt{}$dividend; [II. ax. 18

i.e., if there be a factor larger than $\sqrt{}$dividend, there is also a factor smaller than $\sqrt{}$dividend,

which is impossible, since all factors smaller than $\sqrt{}$dividend have already been tried, and have failed.

Hence every composite number has some factor smaller than its own square root; and if a number have no such factor then it is known to be prime.

E.g., of 11 908 710, 2 is a successful divisor once, 3 twice, 5 once, 11 once, 23 once, and the square root of the quotient, 523, is smaller than 23;

$\therefore$ the prime factors are 2, 3, 3, 5, 11, 23, 523,

and $11\,908\,710 = 2 \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 \cdot 523$.

(*b*) *The number a polynomial that can be reduced to some type-form whose factors are known:*

Reduce the number to the type-form, and write its factors directly, in the form of the factors of the type.

E.g., $x^2 + 2ax + a^2 - 25m^2n^2, \equiv (x+a)^2 - (5mn)^2,$
$= (x + a + 5mn) \cdot (x + a - 5mn).$ [II. 3, 2

(*c*) *The number a polynomial with one letter of arrangement:*

Find the h. c. msr. of the coefficients, and divide by it.

By trial find a polynomial factor of degree not higher than half the degree of the polynomial.

Try no factor unless its first and last coefficients measure the first and last coefficients of the number, respectively.

Try no factor unless its value measures that of the polynomial when the letters have convenient integral values given to them.

If all the coefficients in the polynomial be positive, try no factor whose first and last coefficients are not both positive.

For no integer or simple literal monomial can measure a polynomial unless it measures every term of it. Q. E. D.

And if there be a factor whose degree is higher than half the degree of the polynomial,

then its co-factor is of degree lower than half the degree of the polynomial, [II. th. 5 cr. 5

i.e., lower than the degree of the factor tried,

and the lower factor, not the higher, is best sought. Q. E. D.

And $\because$ the $\left\{\begin{matrix}\text{first}\\\text{last}\end{matrix}\right.$ term of the dividend is the $\left\{\begin{matrix}\text{first}\\\text{last}\end{matrix}\right.$ term of the divisor multiplied by the $\left\{\begin{matrix}\text{first}\\\text{last}\end{matrix}\right.$ term of the quotient,

$\therefore$ every entire measure of the dividend has its $\left\{ \begin{matrix} \text{first} \\ \text{last} \end{matrix} \right.$ term a measure of the $\left\{ \begin{matrix} \text{first} \\ \text{last} \end{matrix} \right.$ term of the dividend. Q. E. D.

And $\because$ if there be an entire measure of the polynomial, the co-factor is then entire, [ax. 8

$\therefore$ whenever the letters have integral values,

then the value of the co-factor is an integer, [II. ax. 23

i.e., the value of the factor then measures the value of the polynomial. Q. E. D.

The last clause of the rule is based on principles stated later.

E.g., to factor $40ax^2 + 130axy + 75ay^2$:

$\because$ a is a common factor, and 5 the h. c. msr. of 40, 130, and 75,

$\therefore$ the expression is resolved into the three factors

$5,\ a,\ 8x^2 + 26xy + 15y^2$,

wherein 1, 2, 4, 8 are the measures of 8, and 1, 3, 5, 15, of 15;

and $\because$ all the coefficients are positive,

$\therefore$ the possible measures of $8x^2 + 26xy + 15y^2$, on its face, are:

$$\begin{array}{llll} x+y, & 2x+y, & 4x+y, & 8x+y, \\ x+3y, & 2x+3y, & 4x+3y, & 8x+3y, \\ x+5y, & 2x+5y, & 4x+5y, & 8x+5y, \\ x+15y, & 2x+15y, & 4x+15y, & 8x+15y. \end{array}$$

In $8x^2+26xy+15y^2$ and in these sixteen possible measures

put $x=1$ and $y=1$;

then $8x^2+26xy+15y^2=49$, whose measures are 1, 7, and 49,

and only $4x+3y, =7$, and $2x+5y, =7$, pass this test;

and $4x+3y$ and $2x+5y$ are found by actual multiplication or division to be the factors sought.

So to factor P, $\equiv 7x^3 - 30x^2 + 62x - 45$:

The only possible linear factors, on its face, are

$$\begin{array}{llllll} x\pm 1, & x\pm 3, & x\pm 5, & x\pm 9, & x\pm 15, & x\pm 45, \\ 7x\pm 1, & 7x\pm 3, & 7x\pm 5, & 7x\pm 9, & 7x\pm 15, & 7x\pm 45. \end{array}$$

In $7x^3-30x^2+62x-45$, and in these twenty-four possible factors, put $x=1$;

then $P = -6$, and the only possible factors of it are

$$x+1, =2; \quad x-3, =-2; \quad x+5, =6;$$
$$7x-1, =6; \quad 7x-5, =2; \quad 7x-9, =-2.$$

So put $x = 2$;

then $P = 15$, and out of the six possible factors above the only ones still possible are

$$x+1, =3; \quad x-3, =-1; \quad 7x-9, =5;$$

then $\because$ of these three possible factors $7x-9$ succeeds, and gives x^2-3x+5 for quotient, and the others fail,

$\therefore$ x^2-3x+5 is prime;

and $7x-9$, x^2-3x+5 are the factors sought.

NOTE. For further discussion of this case see

(*d*) *The number a polynomial; several letters of arrangement:*

Arrange the number as to the powers of any one of the letters (preferably that one whose powers are most numerous), *and unite all terms having any same power of this letter into a complex term. Find the h. c. msr. of the coefficients of the different powers of the letter of arrangement, and take it out as a factor of the polynomial; then the co-factor has no prime measures free from this letter.*

Arrange the polynomial, or the co-factor just found, as to any other letter, and proceed as before, and so on for all the letters; of the co-factor left, the prime factors, if any, will each contain all the letters, and can only be found by trial; but:

Try no factor of more than half the degree of this co-factor as to any letter or letters;

Try no factor that will not measure this co-factor if any one or more of its letters be made zero.

If the polynomial be symmetric as to any of its letters, try no factor that is not either symmetric as to those letters, or one of a set of possible factors that together are symmetric.

So, if the polynomial be partially symmetric as to any letters, (*i.e.*, if for some interchanges among those letters its value would be unchanged,) *try only those factors which, singly or in groups, are likewise either symmetric or partially symmetric.*

E.g., to factor $2x^3 + 6x^2y + 4xy^2 - 3x^2z + xyz + 2y^2z - 8xz^2 - yz^2 - 3z^3$:

$\because$ 2, the coefficient of x^3, is prime to $6y-3z$, the coefficient of x^2, [inspection

$\therefore$ there is no entire measure free from x.

So $\because$ the coefficient of z^3 is prime to that of z^2, [inspection

$\therefore$ there is no entire measure free from z.

But $\because$ the coefficients of y^2, y, y^0 have a h. c. msr. $2x+z$,

$\therefore$ $2x+z$ is a factor of the polynomial,

and the co-factor is $x^2 + 3xy + 2y^2 - 2xz - yz - 3z^2$,

whereof every factor has all the letters, but reduces to

a factor of $2y^2 - yz - 3z^2$, $= \overline{y+z} \cdot \overline{2y-3z}$, when $x = 0$,

a factor of $x^2 - 2xz - 3z^2$, $= \overline{x+z} \cdot \overline{x-3z}$, when $y = 0$,

a factor of $x^2 + 3xy + 2y^2$, $= \overline{x+y} \cdot \overline{x+2y}$, when $z = 0$;

and $\because$ the trinomials $x+y+z$, $x+2y-3z$, and no others, fulfil these conditions, and are found by trial to succeed,

$\therefore$ the factors of the given polynomial are $2x+z$, $x+y+z$, $x+2y-3z$. Q.E.F.

So, to factor $x^2 - 2xy + y^2 - 2xz - 2yz + z^2$:

$\because$ $-2y-2z$ and $y^2 - 2yz + z^2$, the coefficients of x and of x^0, are prime to each other,

$\therefore$ there is no entire measure free from x;

so there is no entire measure free from y,

and none free from z;

$\therefore$ every factor has all the letters, but reduces to

a factor of $y^2 - 2yz + z^2$, *i.e.* to $\pm(y-z)$, when $x = 0$,

a factor of $x^2 - 2xz + z^2$, *i.e.* to $\pm(x-z)$, when $y = 0$,

a factor of $x^2 - 2xy + y^2$, *i.e.* to $\pm(x-y)$, when $z = 0$;

and $\because$ no trinomial fulfils all three conditions,

$\therefore$ the given polynomial is a prime.

Or, $\because$ the given polynomial is symmetric as to x, y, z,

$\therefore$ the factors, if any, must be symmetric as to x, y, z, either as a set, or singly;

but $\because$ such a set would consist of at least three factors,

and $\because$ this polynomial, being of the second degree, can have but two factors,

$\therefore$ the factors are not symmetric as a set.

And $\because$ there can be no single symmetric factor except $x+y+z$,

and $\because$ $x^2-2xy+y^2-2xz-2yz+z^2 \neq (x+y+z)^2$,

$\therefore$ there are no factors symmetric singly,

$\therefore$ the polynomial is a prime.

NOTE 1. The proofs for the rule in case (d) are substantially the same as those given in case (c).

NOTE 2. The work is often aided by introducing new letters of arrangement, as to which the polynomial is more simple, or is homogeneous.

E.g., to factor $\text{P}, \equiv 6x^{10}y^5 - 20x^8y^4z^{\frac{1}{2}} + 25x^6y^3z - 8x^4y^2z^{\frac{3}{2}}$
$- 8x^2yz^2 + 6z^{\frac{5}{2}}$.

Let $u \equiv x^2y$, $v \equiv z^{\frac{1}{2}}$, and seek the factors of
$\text{P}, = 6u^5 - 20u^4v + 25u^3v^2 - 8u^2v^3 - 8uv^4 + 6v^5$.

Try no factors except of the form $\text{A}u + \text{B}v$, or $\text{C}u^2 + \text{D}uv + \text{E}v^2$, wherein A, B, C, E are measures of 6, and the value of the proposed factor is a measure of P when for u and v are put any convenient integers.

When $u, v = 1, 1$, then $\text{P} = 1$, and $\text{A}+\text{B}$ and $\text{C}+\text{D}+\text{E}$, measures of P, each $= 1$.

When $u, v = 1, 2$, then $\text{P} = 223$, a prime, and $\text{A}+2\text{B}$ and $\text{C}+2\text{D}+4\text{E}$, measures of P, each of them $= 223$, which is manifestly larger than the other conditions permit, or else $= 1$.

When $u, v = 2, 1$, then $\text{P} = 30$, and $2\text{A}+\text{B}$ and $4\text{C}+2\text{D}+\text{E}$, measures of P, are measures of 30.

But $\because$ no integers A, B, measures of 6, satisfy all these conditions,

$\therefore$ there is no measure of the form $\text{A}u + \text{B}v$.

And $\because$ the only integers C, D, E that satisfy them are 2, -4, 3,

and $\because$ $2u^2 - 4uv + 3v^2$ is found on trial to measure P, and the quotient is $3u^3 - 4u^2v + 2v^3$;

$$\therefore \text{P} = (2u^2 - 4uv + 3v^2)\cdot(3u^3 - 4u^2v + 2v^3)$$
$$= (2x^4y^2 - 4x^2yz^{\frac{1}{2}} + 3z)\cdot(3x^6y^3 - 4x^2yz^{\frac{1}{2}} + 2z^{\frac{3}{2}}).$$

So to factor P, $\equiv A - Bx + Cx^2 - Dx^3 + \cdots$, wherein A, B, $\cdots$ are positive:

Let $-y \equiv x$, then P becomes $A + By + Cy^2 + Dy^3 + \cdots$, whose factors are often more easily found.

So to factor P, $\equiv 36x^4 - 25x^2 + 4$:

Let $u \equiv x^2$, $v \equiv 1$, then P becomes $36u^2 - 25uv + 4v^2$, whose factors are $4u - v$, $9u - 4v$;

i.e., $4x^2-1, 9x^2-4, =(2x+1)\cdot(2x-1), (3x+2)\cdot(3x-2)$.

$\therefore$ $P = (2x+1)\cdot(2x-1)\cdot(3x+2)\cdot(3x-2)$.

NOTE 3. A polynomial may often be resolved into surd or imaginary factors.

E.g., $x - y = (x^{\frac{1}{2}} + y^{\frac{1}{2}})\cdot(x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

So $2x-3y = (\sqrt{2x}+\sqrt{3y})\cdot(\sqrt{2x}-\sqrt{3y})$

$= (\sqrt{2x}+\sqrt{3y})\cdot(\sqrt[4]{2x}+\sqrt[4]{3y})\cdot(\sqrt[4]{2x}-\sqrt[4]{3y})$

$= \cdots$

So $x^2+1 = (x+\sqrt{-1})\cdot(x-\sqrt{-1})$.

§ 8. EXAMPLES.

§§ 5, 6. PROBS. 1, 2.

$\cdots$ 12. Find the h. c. msr. and l. c. mlt. of:

1. $x-1, x^2-1$; $x-2, x^2-4$; $3(x^3-a^2x), 4(x^2+ax), 5(x^4-a^4)$.
2. $1-x^2, (1+x)^2$; $1-2x, 1-4x^2, 1-8x^3, 1-16x^4, 1-32x^5$.
3. x^2+2x-3, x^3-7x^2+6x; $x^5+x^4+x^3+x^2+x+1, x^2-x+1$.
4. $4+5x+x^2,\ 8-2x-x^2,\ 12+7x+x^2,\ 20+x-x^2$.
5. $529(x^2+x-6),\ 782(2x^2+7x+3),\ 935(2x^2-3x-2)$.
6. $nx^3+3nx^2y-2nxy^2-2ny^3,\ 4mx^3+mx^2y-2mxy^2-3my^3$.
7. $x^4-px^3+(q-1)x^2+px-q,\ x^4-qx^3+(p-1)x^2+qx-p$.
8. $x^3+(4a+b)x^2+(3a^2+4ab)x+3a^2b$,
   $x^3+(2a-b)x^2-(3a^2+2ab)x+3a^2b$.
9. $a^3e^{2x}+e^{2x}-a^3-1,\ (a-2+a^{-1})\cdot(e^x-2+e^{-x})$.
10. $x^2+y^2+z^2+2(xy+yz+zx),\ (\frac{1}{2}x+\frac{1}{2}y+z)^2-(\frac{1}{2}x+\frac{1}{2}y)^2$.
11. $x^{-3}+\frac{3}{4}x^{-2}+\frac{3}{4}x^{-1}+1,\ \frac{1}{4}x^{-2}-\frac{1}{4}$; $\dfrac{x^4-y^4}{x^2-2xy+y^2}, \dfrac{x^2+xy}{x-y}$.

12. $$\frac{\begin{array}{r|l} x^2+a & x-ab \\ -b & \end{array}}{\begin{array}{r|r|l} x^3+2a & x^2-\ 6ab & x-24abc \\ -3b & +\ 8ac & \\ +4c & -12bc & \end{array}}, \qquad \frac{\begin{array}{r|l} x^2+a & x-ac \\ -c & \end{array}}{\begin{array}{r|r|l} x^3-2a & x^2+\ 6ab & x+24abc \\ -3b & -\ 8ac & \\ +4c & -12bc & \end{array}}.$$

... 15. Reduce to lowest terms by means of the h. c. msrs. of their numerators and denominators:

13. $$\frac{x^2-6x+5}{7x^2-12x+5}; \quad \frac{1+3x-4x^2-12x^3}{8x^3-4x^2-2x+1}; \quad \frac{1+x^2+25x^4}{1+3x-15x^2-25x^4}.$$

14. $$\frac{x^{3a}+x^{2a}+x^a+1}{1-x^a+x^{2a}-x^{3a}}; \quad \frac{x^4-a^4}{a^3+a^2x-ax^2-x^3}; \quad \frac{a^6+x^6}{x^{10}+a^{10}}.$$

15. $$\frac{x^{-2}+11x^{-1}+30}{9x^{-3}+53x^{-2}-9x^{-1}-18}; \quad \frac{xy^{-1}+2+x^{-1}y}{xy^{-2}+x^{-2}y}; \quad \frac{x^5+y^{-5}}{x^7+y^{-7}}.$$

... 20. Reduce to lowest common denominator, by means of the l. c. mlts. of the given denominators, and add:

16. $$\frac{1}{2(a+x)}, \quad \frac{3}{4(a-x)}; \quad \frac{5}{6(a^2+x^2)}, \quad \frac{7}{8(a^2-x^2)}, \quad \frac{9}{10(a+ax+x^2)}.$$

17. $$\frac{x^3+y^3}{x^3-y^3}, \quad \frac{x^3-y^3}{x^3+y^3}, \quad \frac{x^2+xy+y^2}{x^2+y^2}, \quad \frac{x^2-xy+y^2}{x^2-y^2}, \quad \frac{x-y}{x+y}, \quad \frac{x+y}{x-y}.$$

18. $$\frac{1}{4x^3(x+y)}, \quad \frac{1}{2x^2(x^2+y^2)}, \quad \frac{1}{4x^3(x-y)}, \quad \frac{1}{2x^2(x^2-y^2)}.$$

19. $$\frac{a+b}{a-b}, \quad \frac{a-b}{a+b}, \quad \frac{a^2+b^2}{a^2-b^2}, \quad \frac{a^2-b^2}{a^2+b^2}, \quad \frac{a^3+b^3}{a^3-b^3}, \quad \frac{a^3-b^3}{a^3+b^3}.$$

20. $$\frac{\begin{array}{r|l} x^2-c & x-cd \\ +d & \end{array}}{\begin{array}{r|l} x^2+a & x-ab \\ -b & \end{array}}, \quad \frac{\begin{array}{r|l} x^2+b & x+bd \\ +d & \end{array}}{\begin{array}{r|l} x^2+a & x+ac \\ +c & \end{array}}, \quad \frac{\begin{array}{r|l} x^2+b & x-bc \\ -c & \end{array}}{\begin{array}{r|l} x^2+a & x-ad \\ -d & \end{array}},$$

$$\frac{\begin{array}{r|l} x^2-a & x-ad \\ +d & \end{array}}{\begin{array}{r|l} x^2-b & x-bc \\ +c & \end{array}}, \quad \frac{\begin{array}{r|l} x^2-a & x+ac \\ -c & \end{array}}{\begin{array}{r|l} x^2-b & x+bd \\ -d & \end{array}}, \quad \frac{\begin{array}{r|l} x^2-a & x-ab \\ +b & \end{array}}{\begin{array}{r|l} x^2+c & x-cd \\ -d & \end{array}}.$$

§ 7. PROB. 3.

21. Factor, or prove to be prime:
30; 37; 72; 120; 323; 367; 1331; 1683; 8279; 15625.

22. Make a table of the prime numbers from 0 to 400.

NOTE. The work is aided by arranging the *odd* numbers 1, 3, 5, 7, 9, ⋯, 399 upon paper ruled in squares, and marking off as composite every 3d of them, beginning with 3^2, every 5th 5^2, and so on. The multiples of any prime, P, thus marked off, have a common difference, 2 P, and often lie in convenient diagonal lines. All the multiples of P thus got from one another may be tested by merely testing the highest of them by division. Why are the small primes most frequent?

23. Use the above table to factor 9991, or to prove it prime.

24. Tabulate the prime factors of the numerals 1, 2, 3, ⋯, 100.

⋯ 46. Factor, or prove to be prime:

25. $\overline{x+1}^2 + \overline{a+3} \cdot \overline{x+1} + 3a$; $x^2 + 2xy + y^2 + 5x + 5y + 6$.

26. $x^2 + y^2 + z^2 - 2xy \pm 2xz \mp 2yz$; $x^2 + y^2 - z^2 \pm 2xy$.

27. $2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$, $= 4a^2b^2 - (a^2 + b^2 - c^2)^2$.

28. $a^2 + 4b^2 + 9c^2 + \cdots + 4ab + 6ac + \cdots + 12bc + \cdots + \cdots$

29. $4a^2b^{-4} + 12ab^{-2}c^3d^{-4} + 9c^6d^8 - 16m^{10}n^{-12}$
$\mp 40m^5n^{-6}p^7q^{-8} - 25p^{14}q^{-16}$.

30. $e^{2x} - e^{-2x}$; $e^{2x} \pm 2 + e^{-2x}$.

31. $(a - a^{-1})^8 - (b - b^{-1})^8$; $a^8 - 256b^{-8}$; $(a + x)^6 - (a - x)^6$.

32. $a^2x^2 - 3a^3x + 2a^4$; $a^3 - a^2x - 6ax^2$; $12a^4 + a^2x^2 - x^4$.

33. $x^3 + y^3 + 3xy(x + y)$; $m^3 - n^3 - m(m^2 - n^2) + n(m - n)^2$.

34. $a^2 - ab - 2(ab - b^2) + 3(a^2 - b^2) - 4(a - b)^2$.

35. $a^3 - b^3 - 3ab(a - b)$; $5(x^2 - y^2) + 3(x + y)^2$.

36. $3(x^2 - y^2) - 5(x - y)^2$; $(x + y)^2 + 2(x^2 + xy) - 3(x^2 - y^2)$.

37. $2(a^3 + a^2b + ab^2) - (a^3 - b^3)$; $a^4 - b^4 + (a^2 - b^2)^2$.

38. $2x^3y + 5x^2y^2 + 2xy^3$; $6y^4 - 3xy^3 - 9x^2y^2$; $6x^4 + x^2y - 12y^2$.

39. $a^4x^2 + a^3x - a^2$; $6b^2x^2 - 7bx^3 - 3x^4$; $6x^3 + 11x^2 + 9x - 35$.

40. $6x^3 - 11x^2 + 9x + 35$; $6x^3 - 11x^2 + 9x + 34$.

41. $x^4 + (a - b + c - d)x^3 + (-ab + ac - ad - bc + bd - cd)x^2$
$+ (-abc + abd - acd + bcd)x + abcd$.

42. $3x^4 - 17x^3 + 38x^2 - 23x + 9$; $5x^4 - 18x^3 + 17x^2 - 18x + 5$;
$15x^4 + 8x^3y - 32xy^3 - 15y^4$. [two trinomial factors

43. $abx^2 + a^2x + b^2x + ab$; $a^3x^3 + b^3y^3 + c^3z^3$; $a^3x^3 - b^3y^3 + c^3z^3$.

44. $39(a^6+a^3b^3)x^4y^2+78(a^6+2a^3b^3+b^6)x^3y^3+156(a^3b^3+b^6)x^2y^4$.

45. $3a^2+6abz-4acz-8bcz^2$; $3a^2-6abz+4acz-8bcz^2$.

46. $a^3x^3+2a^2bx^2+2ab^2x+b^3$; $a^4x^4+a^2b^2x^2+b^4$.

47. Factor $45x^3+83x^2y-100xy^2-49y^3$.

NOTE. Only measures of the form $\text{A}x+\text{B}y$ need be tried, and here A, B, being measures of 45, 49, are odd integers;
but $\because$ A + B, the value of the proposed measure when x, y each $=1$, is an even integer and cannot measure the corresponding value of the polynomial, the odd integer $45+83-100-49$;
$\therefore$ the polynomial is a prime.

48. Show in like manner which are primes of:
$7x^3-10x^2+9x+5$, $7x^3-25x^2+11x+3$,
$5x^2+17x+3$, $a^2\pm ab+b^2$, $a^3\pm a^2b+ab^2\pm b^3$;
and generalize for any trinomial or quadrinomial whose first and last coefficients and one other are odd.

49. Resolve into three symmetric factors:
$$-2x^3-2y^3-2z^3+5x^2y+5y^2z+5z^2x-xy^2-yz^2-zx^2+2xyz.$$

50. If $\text{F}(x)$ be any entire function of x, prove that $\text{F}(x)-\text{F}(a)$ is measured by $x-a$, and hence that
If $\text{F}(a)=0$, then $\text{F}(x)$ is measured by $x-a$. Hence factor:
$(x^3+2x+3)\cdot(a^3+a)-(a^3+2a+3)(x^3+x)$.

51. Prove that $x^qy^r+y^qz^r+z^qx^r-x^ry^q-y^rz^q-z^rx^q$ is measured by $(x-y)\cdot(y-z)\cdot(z-x)$ if q and r be any positive integers. Hence factor:
$x^2y+y^2z+z^2x-xy^2-yz^2-zx^2$;
$x^3y+y^3z+z^3x-xy^3-yz^3-zx^3$;
$x^3y^2+y^3z^2+z^3y^2-x^2y^3-y^2z^3-z^2x^3$.

52. Prove that the $\left\{\begin{matrix}\text{h. c. msr.}\\ \text{l. c. mlt.}\end{matrix}\right.$ of two or more numbers is the reciprocal of the $\left\{\begin{matrix}\text{l. c. mlt.}\\ \text{h. c. msr.}\end{matrix}\right.$ of their reciprocals.

53. Prove that the $\left\{\begin{matrix}\text{numerator}\\ \text{denominator}\end{matrix}\right.$ of a simple fraction in its lowest terms is the l. c. mlt. of $\left\{\begin{matrix}\text{the fraction}\\ \text{the reciprocal fraction}\end{matrix}\right.$ and 1.

IV. PERMUTATIONS AND COMBINATIONS.

§ 1. DEFINITIONS.

THE different orders in which several things or *elements* can be put, are their *permutations* or *arrangements;* the different groups that can be made of them, without regard to order, are their *combinations.* Two permutations are different when either the things themselves are different or their order of arrangement is different; but two combinations are different only when at least one of the things contained in one of them is not found in the other.

E.g., *ab*, *ba*, *ac*, *ca*, *bc*, *cb* are the six permutations of *a*, *b*, *c*, taken two at a time;

but *ab* and *ba* are the same combination, *ac* and *ca* are the same, and *bc* and *cb* are the same,

and, in all, there are but three distinct combinations.

So *abc*, *bac*, *acb*, *cab*, *bca*, *cba* are the six permutations of *a*, *b*, *c*, taken all together;

but there is only one combination, in whatever order the three things are taken.

So of four things, *a*, *b*, *c*, *d*, there are four combinations, taken three at a time: *abc*, *abd*, *acd*, *bcd*,

and of each of them can be made six permutations, as above —twenty-four in all.

§ 2. PERMUTATIONS.

PROB. 1. TO FORM THE SEVERAL PERMUTATIONS OF n THINGS, ALL DIFFERENT, TAKEN 1, 2, 3, ··· AT A TIME.

To each of the n *things in turn, annex each of the* (n−1) *things remaining; the results are the couplets.*

To each of the couplets in turn, annex each of the (n−2) *things remaining; the results are the triplets.*

To each of the triplets in turn, annex each of the (n−3) *things remaining; the results are the fours; and so on.*

E.g., of the four things a, b, c, d the permutations are

single things:

	a,	b,	c,	d;
couplets:				
	ab, ac, ad,	ba, bc, bd,	ca, cb, cd,	da, db, dc;
triplets:				
	abc, abd,	bac, bad,	cab, cad,	dab, dac,
	acb, acd,	bca, bcd,	cba, cbd,	dba, dbc,
	adb, adc,	bda, bdc,	cda, cdb,	dca, dcb;
fours:				
	$abcd$, $abdc$,	$bacd$, $badc$,	$cabd$, $cadb$,	$dabc$, $dacb$,
	$acbd$, $acdb$,	$bcad$, $bcda$,	$cbad$, $cbda$,	$dbac$, $dbca$,
	$adbc$, $adcb$,	$bdac$, $bdca$,	$cdab$, $cdba$,	$dcab$, $dcba$.

THEOR. 1. *The process of* Pr. 1 *gives all the possible permutations, and no two of them so formed are alike.*

The number of permutations of n *things, all different,*

$$\textit{taken}\left\{\begin{array}{l}1\\2\\3\\\cdots\\r\end{array}\right. \textit{at a time, is}\left\{\begin{array}{l}\mathrm{n}.\\\mathrm{n}\cdot(\mathrm{n}-1).\\\mathrm{n}\cdot(\mathrm{n}-1)\cdot(\mathrm{n}-2).\\\cdots\quad\cdots\quad\cdots\quad\cdots\\\mathrm{n}\cdot(\mathrm{n}-1)\cdot(\mathrm{n}-2)\cdots(\mathrm{n}-\mathrm{r}+1).\end{array}\right.$$

Let $a, b, c, d, \cdots h, k, l \equiv$ any n things all different, and let $\mathrm{P}_1 n$, $\mathrm{P}_2 n$, $\mathrm{P}_3 n$, $\cdots$ $\mathrm{P}_r n \equiv$ the number of permutations of these n things taken 1 at a time, 2 at a time, 3 at a time, $\cdots$ r at a time.

(*a*) *One at a time.*

For $\because$ of one thing there is one and but one permutation,

$\therefore$ of the n things, taken one at a time, there are n permutations, viz., one for each thing, and no more.

i.e., $\mathrm{P}_1 n = n$. Q.E.D.

(*b*) *Two at a time.*

For to each of the n things in turn, annex each one of the $n-1$ things remaining;

then $ab, ac, ad, \cdots al$ form $(n-1)$ couplets with a first,

$ba, bc, bd, \cdots bl$ form $(n-1)$ couplets with b first,

$ca, cb, cd, \cdots cl$ form $(n-1)$ couplets with c first,

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots,$

and $la, lb, lc, \cdots lk$ form $(n-1)$ couplets with l first;

whereof no two are alike, since each one has either a first letter, a second letter, or both, different from every other,

and there are no possible couplets omitted, since every letter in turn is joined, both as first letter and as second letter, with every other letter;

$\therefore$ of the n things, taken two at a time, there are $n \cdot (n-1)$ permutations, and no more;

i.e., $P_2 n = n \cdot (n-1)$. Q.E.D.

(c) *Three at a time.*

For to each of the $n \cdot (n-1)$ couplets in turn, annex each one of the $(n-2)$ things remaining;

then $\because$ *abc*, *abd*, *abe*, $\cdots$ *abl* form $n-2$ triplets with *ab* first,
acb, *acd*, *ace*, $\cdots$ *acl* form $n-2$ triplets with *ac* first,
$\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$,

and *lka*, *lkb*, *lkc*, $\cdots$ *lkh* form $n-2$ triplets with *lk* first,

whereof no two are alike, since each one of them has either the leading couplet, or the letter that follows it, or both, different from every other,

and there are no possible triplets omitted, since every possible couplet, in turn, is followed by every letter not already in it;

$\therefore$ of the n things, taken three at a time, there are $n \cdot (n-1) \cdot (n-2)$ permutations, and no more;

i.e., $P_3 n = n \cdot (n-1) \cdot (n-2)$. Q.E.D.

So $P_4 n = n \cdot (n-1) \cdot (n-2) \cdot (n-3)$,
$P_5 n = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4)$,
$\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$ $\cdots$,

and $P_r n = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$, for any value of r not greater than n. Q.E.D.

NOTE 1. This proof is by induction, but it is of so simple a character that it need not be put in the formal order given in II. § 1, th. 3 $(a-c)$, $\cdots$. The reader may, however, as an exercise, make the statement formal.

NOTE 2. The expressions

$n,\ n \cdot (n-1),\ n \cdot (n-1) \cdot (n-2), \cdots\ n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$

may be severally written in the equivalent forms:

$$\frac{n!}{(n-1)!},\ \frac{n!}{(n-2)!},\ \frac{n!}{(n-3)!},\ \frac{n!}{(n-4)!},\ \cdots\ \frac{n!}{(n-r)!}.$$

COR. 1. *Of* n *things, all different, taken all together, there are* n ! *permutations.*

For $\because$ $n-r+1=1$ when $r=n$,

$$\therefore\ \text{P}_n n = n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$$
$$= 1\cdot 2\cdot 3\cdots n$$
$$= n! \qquad \text{Q. E. D.}$$

NOTE 1. The expression $n!$, hitherto defined as the continued product of the natural numbers $1\cdot 2\cdot 3 \cdots n$ [I. § 8], may have a useful extension.

For $\because$ $n! \equiv 1\cdot 2\cdot 3\cdots n$, and $(n-1)! \equiv 1\cdot 2\cdot 3\cdots(n-1)$,

$\therefore$ $n! = (n-1)!\cdot n$,

$\therefore$ $(n-1)! = n! : n$.

So $(n-2)! = (n-1)! : (n-1)$,

$(n-3)! = (n-2)! : (n-2)$, and so on.

Conceive this relation to hold true for all integers, whether positive, zero, or negative;

then $1! = 2! : 2 = 1$, $0! = 1! : 1 = 1$.

With this explanation the form $n! : (n-r)!$ becomes intelligible when $r=n$, as in Cor. 1, for then

$n! : (n-r)! = n! : 0! = n! : 1 = n!$,

and the result, the value of $\text{P}_n n$, is the same through whichever form it is reached.

NOTE 2. Another and independent proof of Cor. 1 is as follows:

Let a, b, c, $\cdots$ be n things, all different;

then $\because$ of the one thing a, there is one permutation, and but one,

$\therefore$ $\text{P}_1 1 = 1$, which, for conformity with what follows, may be written $1!$ Q. E. D.

Place b in each of the only two possible positions with respect to a, *i.e.* after a and before a, giving ab and ba;

then $\because$ of two things a, b, there are two permutations, and but two,

$\therefore$ $\text{P}_2 = 2$, which may be written $2!$ Q. E. D.

Place c in each of the only three possible positions with respect to a and b in these two couplets ab, ba, giving abc, acb, cab, bac, bca, cba;

then ∵ of three things a, b, c, there are $2 \cdot 3$, $= 6$, permutations, and but 6,

∴ $\mathrm{P}_3 3 = 2\,! \cdot 3 = 3\,!$ Q. E. D.

Place d in each of the only four possible positions with respect to a, b, c in these $3\,!$ triplets;

then ∵ of four things a, b, c, d, there are $3\,! \cdot 4$ permutations,

∴ $\mathrm{P}_4 4 = 3\,! \cdot 4 = 4\,!$ Q. E. D.

So $\mathrm{P}_5 5 = 4\,! \cdot 5 = 5\,!$

$\mathrm{P}_6 6 = 5\,! \cdot 6 = 6\,!$, and so on.

∴ $\mathrm{P}_n n = (n-1)\,! \cdot n = n\,!$, n any positive integer. Q. E. D.

This note embodies a rule for forming the permutations of n things taken all together. The reader may state it, and illustrate it by the permutations of a, b, c and of a, b, c, d.

Cor. 2. $\mathrm{P}_r(\mathrm{n}+1) = \mathrm{P}_r \mathrm{n} + \mathrm{r} \cdot \mathrm{P}_{r-1} \mathrm{n}$.

For ∵ $\mathrm{P}_r(n+1) = (n+1) \cdot n \cdot (n-1) \cdots (n-r+2)$,

and ∵ $\mathrm{P}_r n + r \cdot \mathrm{P}_{r-1} n = n \cdot (n-1) \cdots (n-r+2) \cdot (n-r+1)$
$+ n \cdot (n-1) \cdots (n-r+2) \cdot r$
$= n \cdot (n-1) \cdots (n-r+2) \cdot (n+1)$,

∴ $\mathrm{P}_r(n+1) \quad = \mathrm{P}_r n + r \cdot \mathrm{P}_{r-1} n$. Q. E. D.

Note. Another and independent proof of Cor. 2 is as follows:

Let a, b, c, $\cdots k$ be any n things, all different, and l another;

then ∵ $\mathrm{P}_r n \equiv$ the number of permutations of the n things, $a \cdots k$, taken r at a time,

and ∵ $\mathrm{P}_{r-1} n \equiv$ the number of permutations of the n things, $a \cdots k$, taken $r-1$ at a time,

and ∵ no permutations of the $n+1$ things, $a \cdots l$, taken r at a time, can be formed except those of the n things, $a \cdots k$, taken r at a time, and those of the n things, $a \cdots k$, taken $r-1$ at a time, with the new thing l placed in each of the r possible positions therein,

∴ $\mathrm{P}_r(n+1) = \mathrm{P}_r n + r \cdot \mathrm{P}_{r-1} n$.

This note embodies a new rule for forming the permutations of n things taken r at a time. The reader may state it. It also serves to *interpret* the formula, *i.e.*, to show what property of the arrangements the formula expresses.

Many algebraic results derive their chief interest from thus admitting proofs of two kinds, by interpretation, and by more formal methods; and the two lines of proof often curiously correspond. The reader should therefore accustom himself to looking for such interpretations. He will find many of them connected with the subject of permutations and combinations: *e.g.*, [th. 3 cr. 1 nt., cr. 2 nt.].

THEOR. 2. *If* n *things, whereof* p *things are alike,* q *things alike,* r *things alike,* ⋯ *be taken all together, there are* $\dfrac{\mathrm{n}!}{\mathrm{p}!\cdot\mathrm{q}!\cdot\mathrm{r}!\cdots}$ *different permutations of them.*

E.g., if there be two 5's, three 6's, and four 7's, then of these nine digits $\dfrac{9!}{2!\cdot 3!\cdot 4!}$, $= 1260$, different nine-figured numbers can be formed.

For take the n things in the several positions they hold in any one of their permutations, and let p things alone change places, while the $n-p$ things remaining stand fast;

then if the p things be all different, $p!$ permutations are got;
but if the p things be all alike, only one permutation is got.

So for every set of positions in which the p things stand.

∴ there are $p!$ times more permutations of the n things when any p of them are all different than when those p things are alike;

i.e., $\mathrm{P}_n n_{\text{all different}} = p!\cdot \mathrm{P}_n n_{p\text{ alike}}$.

So $\mathrm{P}_n n_{\text{all different}} = p!\cdot q!\cdot r!\cdots \cdot \mathrm{P}_n n_{p\text{ alike, } q\text{ alike, } r\text{ alike, } \ldots}$,

But ∵ $\mathrm{P}_n n_{\text{all different}} = n!$

∴ $\mathrm{P}_n n_{p\text{ alike, } q\text{ alike, } r\text{ alike, } \ldots} = \dfrac{n!}{p!\cdot q!\cdot r!\cdots}$. Q. E. D.

In particular:

$$\mathrm{P}_n n_{n\text{ alike}} = 1,$$

$$\mathrm{P}_n n_{n-1\text{ alike}} = n,$$

$$\mathrm{P}_n n_{2\text{ alike, } n-2\text{ alike}} = \frac{n!}{2!\cdot(n-2)!},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots,$$

$$\mathrm{P}_n n_{r\text{ alike, } n-r\text{ alike}} = \frac{n!}{r!\cdot(n-r)!}.$$

§ 3. COMBINATIONS.

Prob. 2. To find the several combinations of n things, taken 1, 2, 3, ⋯ at a time.

To each of the n *things, in turn, annex each of the things that follow it; the results are the couplets.*

To each of the couplets, in turn, annex each of the things that follow all its elements; the results are the triplets; and so on.

E.g., of the four things a, b, c, d the combinations are single things:

a, b, c, d;

couplets:

ab, ac, ad, bc, bd, cd;

triplets:

abc, abd, acd, bcd.

So of 210, the prime factors are: 2, 3, 5, 7;
the 2d degree factors: $2 \cdot 3$, $2 \cdot 5$, $2 \cdot 7$, $3 \cdot 5$, $3 \cdot 7$, $5 \cdot 7$;
the 3d degree factors: $2 \cdot 3 \cdot 5$, $2 \cdot 3 \cdot 7$, $2 \cdot 5 \cdot 7$, $3 \cdot 5 \cdot 7$.

By this process are formed all the possible sets in which the several things are arranged in their normal order; viz.: every such couplet possible, and from these couplets, every such triplet possible, and so on; and the sets so formed embrace every possible combination.

For if any sets were formed with the order of the letters changed, such sets, though different permutations, would be but the same combinations repeated. [§ 1

Theor. 3. *The number of combinations of* n *things, all different, taken* r *at a time, is* $\dfrac{n!}{r! \cdot (n-r)!}$.

For take any r of the n things, and let them change places, while the $n-r$ things remaining stand fast;
then ∵ of these r things, there are $r!$ permutations,
but only one combination,
and so for every set of r things;
∴ there are $r!$ times more permutations of n things, taken r at a time, than there are combinations,
i.e., $P_r n = r! \cdot C_r n$,

wherein $C_r n \equiv$ the number of combinations of n things taken r at a time.

But $\because$ $P_r n = \dfrac{n!}{(n-r)!}$,

$\therefore$ $C_r n = \dfrac{n!}{r!\cdot(n-r)!}$. Q. E. D.

NOTE 1. A useful way of writing the formulae is:

$$C_1 n = \frac{n}{1!}, \quad C_2 n = \frac{n(n-1)}{2!},$$

$$C_3 n = \frac{n(n-1)(n-2)}{3!}, \cdots C_r n = \frac{n(n-1)\cdots(n-r+1)}{r!},$$

or this: $C_1 n = \dfrac{n}{1}, \quad C_2 n = \dfrac{n}{1}\cdot\dfrac{n-1}{2}, \; = C_1 n\cdot\dfrac{n-1}{2},$

$$C_3 n = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} = C_2 n\cdot\frac{n-2}{3}, \cdots$$

$$C_r n = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{n-r+1}{r} = C_{r-1} n\cdot\frac{n-r+1}{r},$$

wherein the successive terms of the series are got by multiplying the preceding terms by fractions of the form $\dfrac{n-k}{k+1}$, whose numerators decrease, and denominators increase, by one, at every step.

COR. 1. $C_r n = C_{n-r} n = P_n n_{r \text{ alike},\ n-r \text{ alike}}$.

For $\because$ each of them $= \dfrac{n!}{r!\,(n-r)!}$,

$\therefore$ they are equal to each other. [II. ax. 1

In particular:

$$C_0 n = C_n n \;= 1, \;= P_n n_{\,n \text{ alike}}.$$

$$C_1 n = C_{n-1} n = n, \;= P_n n_{\,n-1 \text{ alike}}.$$

$$C_2 n = C_{n-2} n = \frac{n!}{2!\,(n-2)!}, \;= P_n n_{\,2 \text{ alike},\ n-2 \text{ alike}}.$$

$$C_3 n = C_{n-3} n = \frac{n!}{3!\,(n-3)!}, \;= P_n n_{\,3 \text{ alike},\ n-3 \text{ alike}}.$$

NOTE. Another and independent proof of Cor. 1 is from its interpretation, as follows:

$\because$ for every set of r things taken out of n things there is left one set of $n-r$ things, and but one,

∴ the number of combinations of any n things, when taken r at a time, and when taken $n-r$ at a time, is the same. Q. E. D.

So if any same n things, whereof r are alike and $n-r$ alike, be permuted in any same n places,

then ∵ when any two of the permutations are $\left\{\begin{matrix}\text{different,}\\ \text{alike,}\end{matrix}\right.$ the r things occupy $\left\{\begin{matrix}\text{a different}\\ \text{the same}\end{matrix}\right.$ combination of places,

∴ the number of permutations, $\mathrm{P}_n n_{r\text{ alike},\ n-r\text{ alike}}$ equals the number of combinations, $\mathrm{C}_r n$. Q. E. D.

Cor. 2. $\mathrm{C}_r(\mathrm{n}+1)=\mathrm{C}_r\mathrm{n}+\mathrm{C}_{r-1}\mathrm{n}$.

For ∵ $\mathrm{C}_r(n+1)=\dfrac{(n+1)\cdot n\cdot(n-1)\cdots(n-r+2)}{r\,!}$,

and ∵ $\mathrm{C}_r n+\mathrm{C}_{r-1}n$

$$=\frac{n\cdot(n-1)\cdots(n-r+1)}{r\,!}+\frac{n\cdot(n-1)\cdots(n-r+2)}{(r-1)\,!}$$

$$=\frac{n\cdots(n-r+2)\cdot(n-r+1)}{r\,!}+\frac{n\cdots(n-r+2)\cdot r}{r\,!}$$

$$=\frac{n\cdot(n-1)\cdots(n-r+2)\cdot(n+1)}{r\,!},$$

∴ $\mathrm{C}_r(n+1)=\mathrm{C}_r n+\mathrm{C}_{(r-1)}n$. Q. E. D.

Note. Another and independent proof of Cor. 2 is as follows:

Let $a, b, c, \cdots k$ be any n things all different, and l another;

then ∵ $\mathrm{C}_r n\equiv$ the number of combinations of the n things $a\cdots k$, taken r at a time,

and ∵ $\mathrm{C}_{r-1}n\equiv$ the number of combinations of the n things $a\cdots k$, taken $r-1$ at a time,

and ∵ no combinations of the $n+1$ things $a\cdots l$, taken r at a time, can be formed except those of the n things $a\cdots k$, taken r at a time, and those of the n things $a\cdots k$, taken $r-1$ at a time and followed by the new thing l,

∴ $\mathrm{C}_r(n+1)=\mathrm{C}_r n+\mathrm{C}_{r-1}n$.

This note embodies a new rule for forming the combinations of n things taken r at a time. The reader may state it. It also serves to interpret the formula, and show what property of the combinations the formula expresses.

THEOR. 4. *If there be* n *things, all different,* p, q, r, ⋯ *be any numbers such that* $p+q+r+\cdots=n$, *then there are* $\frac{n!}{p!\cdot q!\cdot r!\cdots}$ *ways in which these* n *things can be made up into sets, whereof the first set contains* p *things, the second set* q *things, the third set* r *things, and so on.*

E.g., ten soldiers may be formed into three guards, of 2, 3, and 5 men respectively, in $\frac{10!}{2!\cdot 3!\cdot 5!}$, $=2520$, different ways.

For let the first p things constitute the first set, the next q things the second set, and so on, and let the n things change places in every way possible, forming, in all, $n!$ permutations;

then ∵ within each set of p things there are $p!$ permutations, within each set of q things $q!$ permutations, and so on,

and ∵ each of the $p!$ permutations combines with each of the $q!$ permutations, so that each of the double sets gives $p!\cdot q!$ permutations, and so on;

∴ for every way in which the sets are made up there are $p!\cdot q!\cdot r!\cdots$ permutations,

i.e., $\quad P_n n = p!\cdot q!\cdot r!\cdots C_{p,q,r,\ldots} n$;

$$\therefore\ C_{p,q,r,\ldots} n = \frac{n!}{p!\cdot q!\cdot r!\cdots}. \qquad \text{Q. E. D.}$$

NOTE. Expressed in the notation of this theorem,

$$C_r n, = C_{n-r} n, \equiv C_{r,n-r} n = C_{n-r,r} n.$$

COR. 1. *If the number of sets be given, the greatest possible value of* $C_{p,q,r,\ldots} n$ *is when no two of the numbers* p, q, r, ⋯ *differ by more than a unit, one from the other.*

For, if $p > q+1$,

then ∵ $p!\cdot q! \equiv p\cdot(p-1)!\cdot q!$

and $\quad (p-1)!\cdot(q+1)! \equiv (q+1)\cdot(p-1)!\cdot q!$,

$$\therefore\ p!\cdot q! > (p-1)!\cdot(q+1)!$$

$$\therefore\ \frac{n!}{p!\cdot q!\cdot r!\cdots} < \frac{n!}{(p-1)!\cdot(q+1)!\cdot r!\cdots};$$

i.e., $\quad C_{p,q,r,\ldots} n < C_{p-1,q+1,r\ldots} n$,

and $c_{p,q,r,\dots}n$ is not the greatest possible if p exceeds q by more than a unit.

So of any other pair of them.

$\therefore$ $c_{p,q,r,\dots}n$ is greatest when etc. Q. E. D.

In particular: If n be an $\left\{\begin{array}{l}\text{even}\\ \text{odd}\end{array}\right.$ number,

then $c_r n$ is greatest when $r = \left\{\begin{array}{l}\frac{1}{2}n.\\ \frac{1}{2}(n \pm 1).\end{array}\right.$

COR. 2. *There are* $\dfrac{\mathrm{n}!}{\mathrm{a}!\cdot\mathrm{b}!\cdots(\mathrm{p}!)^{\mathrm{a}}\cdot(\mathrm{q}!)^{\mathrm{b}}\cdots}$ *ways of making up* n *things, all different, into a collection of* a *sets of* p *things each,* b *sets of* q *things each, and so on; wherein* $\mathrm{ap} + \mathrm{bq} + \cdots = \mathrm{n}$.

E.g., a boatclub of 10 men can be divided into three pair-oars and a four in $\dfrac{10!}{3!\cdot 1!\cdot(2!)^3\cdot(4!)^6}$, $= 3150$, different ways.

For $\because$ there are $\dfrac{n!}{p!\cdot p!\cdots a \text{ times}\cdot q!\cdot q!\cdots b\text{ times}\cdots}$ ways of making up n things into sets, whereof the first a sets contain p things each, the next b sets contain q things each, $\cdots$, [th.

and $\because$ of these ways, by reason of the permutation of the a sets among themselves, the b sets among themselves, $\cdots$, there are $a!\cdot b!\cdots$ for every way in which the collection of $a + b + \cdots$ sets is made up,

$\therefore$ $\dfrac{n!}{(p!)^a\cdot(q!)^b\cdots} = a!\cdot b!\cdots$ times the number of ways in which the collection can be made up;

$\therefore$ that number is $\dfrac{n!}{a!\cdot b!\cdots(p!)^a\cdot(q!)^b\cdots}$. Q. E. D.

THEOR. 5.' *If there be* n *sets of things, containing* p, q, r, $\cdots$ *things respectively, and if combinations of* n *things be made up by taking one thing from each set, then the number of such combinations is* $\mathrm{p}\cdot\mathrm{q}\cdot\mathrm{r}\cdots$.

For, let the n sets be $a_1, a_2, a_3, \cdots a_p$, $b_1, b_2, b_3, \cdots b_q$, $c_1, c_2, c_3, \cdots c_r$, $\cdots$, and write the first combination $a_1 b_1 c_1 \cdots$;

then while the $b_1 c_1 \cdots$ stand fast, substitute $a_2, a_3, \cdots a_p$ in turn for a_1, thus forming p combinations.

So in each of these, in turn, substitute $b_2, b_3, \cdots b_q$ for b_1, thus forming q combinations from one of them, and $p \cdot q$ combinations from all of them.

So in each of these, in turn, substitute $c_2, c_3, \cdots c_r$, for c_1, thus forming r combinations from one of them, and $p \cdot q \cdot r$ combinations from all of them.

So $\cdots$, thus forming $p \cdot q \cdot r \cdots$ combinations. Q. E. D.

Cor. 1. *If there be a set of* p *things, a set of* q *things, a set of* r *things*, $\cdots$, *there can be made up* $\mathrm{c}_i\mathrm{p} \cdot \mathrm{c}_j\mathrm{q} \cdot \mathrm{c}_k\mathrm{r} \cdots$ *combinations by taking* i *things from the first set*, j *things from the second set*, k *things from the third set, and so on.*

Cor. 2. *With the data of* Cor. 1 *the number of permutations is* $(\mathrm{i}+\mathrm{j}+\mathrm{k}+\cdots)! \cdot \mathrm{c}_i\mathrm{p} \cdot \mathrm{c}_j\mathrm{q} \cdot \mathrm{c}_k\mathrm{r} \cdots$.

Theor. 6. *If there be* n *numbers, all different, and if all possible homogeneous products of the* r*th degree* (combinations with repetition) *be made of them, including their* r*th powers and the products of their* 1*st*, 2*d*, 3*d*, $\cdots$ (r$-$1)*th powers combined in all possible ways, so that there shall be* r *factors in each product, and no more, then the number of such products is*

$$\mathrm{c}_{\mathrm{r},\, with\ repetitions}\mathrm{n}, = \mathrm{c}_\mathrm{r}(\mathrm{n}+\mathrm{r}-1), = \frac{\mathrm{n}(\mathrm{n}+1)\cdots(\mathrm{n}+\mathrm{r}-1)}{\mathrm{r}!}.$$

For, let $a, b, c, \cdots$ be n numbers, all different, and in each of these $\mathrm{c}_{r,\,\text{with repetitions}}\,n$ products let the letters be put in alphabetical order, *e.g.*, $aaa\cdots$, $bdde\cdots$; and then, while the first letter in each product stands fast, let the second letter be replaced by the letter next after it in the alphabet; the third letter, by the letter next but one after it in the alphabet; $\cdots$ the rth letter, by the letter that is $r-1$ steps beyond it in the alphabet, *e.g.*, $aaaa\cdots$ by $abcd\cdots$, $bdde\cdots$ by $befh\cdots$;

then $\because$ each of the $\mathrm{c}_{r,\,\text{with repetitions}}\,n$ products is thus changed into a combination wherein no two elements are alike, and no element is beyond the $(n+r-1)$th letter of the alphabet,

$\therefore$ each product is changed into some one of the $\mathrm{c}_r(n+r-1)$ combinations of r letters, without repetitions, of $(n+r-1)$ letters;

and $\because$ all the combinations so formed are unlike, either in their first letters or in their second letters or $\cdots$, in the same way as are the products from which they were got,

$\therefore$ to each of the products there corresponds a different one of the combinations of $n+r-1$ things taken r at a time without repetitions;

$\therefore$ $c_{r,\text{ with repetitions}}\,n \not> c_r(n+r-1)$.

Again, let the elements of each of the $c_r(n+r-1)$ combinations be put in alphabetical order, and then, while the first element in each combination stands fast, let the second element be replaced by the letter next before it in the alphabet; the third element, by the letter two places before it in the alphabet, and so on;

then $\because$ each of the $c_2(n+r-1)$ combinations thus gives a product wherein no element is beyond the nth letter of the alphabet, and no two letters stand in inverse alphabetical order, though some may be repeated,

$\therefore$ each combination gives one of the $c_{r,\text{ with repetitions}}\,n$ products;

and $\because$ all the combinations so formed are unlike,

$\therefore$ $c_r(n+r-1) \not> c_{r,\text{ with repetitions}}\,n$.

$\therefore$ $c_{r,\text{ with repetitions}}\,n = c_r(n+r-1)$. Q. E. D.

§ 4. EXAMPLES.

§ 2.

1. Find the number of permutations of 10 things, all different, taken 3 at a time; 5 at a time; 7 at a time; all together.

2. Find the number of permutations of 10 things, taken all together, when 3 are alike and 7 alike; when 2 are alike, 3 alike, and 5 alike.

3. In how many different ways can the letters of the continued product a^3b^3 be written? of $a^2b^3c^5$? of $ab^2c^3d^4e^5$?

4. How many permutations can be formed from the word *Cornell? Washington? Constantinople?*

5. In how many ways can 8 men stand in a row? 12 men? 16 men? n men?

6. In how many ways can 8 men sit at a round table? 12 men? 16 men? n men?

7. Of how many things, all different, are there 720 permutations? of how many, all different and taken 3 at a time, are there 210 permutations?

8. How many different permutations, taken three at a time, can be formed from the word *science?* from the word *constitution?*

9. Write out the several permutations and combinations of the 4 digits 1, 2, 3, 4, taken 1 at a time, 2 at a time, 3 at a time, 4 at a time.

10. Find all the factors, prime and composite, of 6; of 30; of 240; of 2310; of 30030; of ab; of abc; of ab^2c^3; of $abcd$; of a^4-x^4.

§ 3.

11. Find the number of combinations of 10 things, all different, taken 3 at a time; 5 at a time; 7 at a time. Show from the example why the number of sets, taken 3 at a time, is the same as the number taken 7 at a time, and why the number taken 5 at a time is largest of all.

12. How many triangles can be formed by joining 3 vertices of a polygon of n sides? how many quadrilaterals by joining 4 vertices? how many pentagons by joining 5 vertices?

13. If a line be cut at 4 points, how many segments are formed? at 6 points? at 9 points? at n points?

14. If there be 4 straight lines in a plane, whereof no two are parallel, and no three meet in a common point, how many triangles are formed? if 5 lines? if 8 lines? if n lines?

15. In how many ways can 10 things be made up into a set of 2, a set of 3, and a set of 5?

16. How many different sums of money can be formed from 1 cent, 1 half-dime, 1 dime, 3 quarter-dollars, 5 dollars?

17. From a party of 6 ladies and 7 gentlemen, how many companies of 4 ladies and 4 gentlemen can be formed? how many sets of 4 couples for a dance?

18. If the number of combinations of n things, taken 4 at a time, be twice the number of permutations, taken 3 at a time, what is the number n?

19. Of the combinations of 8 letters, $a, b, c, \cdots$, taken 4 at a time, how many contain both a and b? a and not b? neither a nor b?

... 25. Show that:

20. $$\mathrm{P}_r(n+1) = \mathrm{P}_r n + r \cdot \mathrm{P}_{r-1}(n-1) + r \cdot (r-1) \cdot \mathrm{P}_{r-2}(n-2) + \cdots + r! \cdot \mathrm{P}_0(n-r+1)$$
$$= \mathrm{P}_r n + \mathrm{P}_1 r \cdot \mathrm{P}_{r-1}(n-1) + \mathrm{P}_2 r \cdot \mathrm{P}_{r-2}(n-2) + \cdots + \mathrm{P}_r r \cdot \mathrm{P}_0(n-r+1).$$

21. $$\mathrm{P}_r(m+n) = \mathrm{P}_r m + \mathrm{P}_1 r \cdot \mathrm{P}_{r-1} m \cdot \mathrm{P}_1 n + \mathrm{P}_2 r \cdot \mathrm{P}_{r-2} m \cdot \mathrm{P}_2 n + \cdots + \mathrm{P}_r n.$$

22. $$\mathrm{C}_r(n+1) = \mathrm{C}_{r-1} n + \mathrm{C}_{r-1}(n-1) + \mathrm{C}_{r-1}(n-2) + \cdots + \mathrm{C}_{r-1}(r-1).$$

23. $$\mathrm{C}_r(m+n) = \mathrm{C}_r m + \mathrm{C}_{r-1} m \cdot \mathrm{C}_1 n + \mathrm{C}_{r-2} m \cdot \mathrm{C}_2 n + \cdots + \mathrm{C}_1 m \cdot \mathrm{C}_{r-1} n + \mathrm{C}_r n.$$

24. $$\begin{aligned}\mathrm{C}_r(m+n+p+\cdots) &= \mathrm{C}_r m + \mathrm{C}_{r-1} m \cdot \mathrm{C}_1 n + \mathrm{C}_{r-1} m \cdot \mathrm{C}_1 p + \cdots \\ &+ \mathrm{C}_{r-2} m \cdot \mathrm{C}_2 n + \mathrm{C}_{r-2} m \cdot \mathrm{C}_2 p + \cdots \\ &+ \mathrm{C}_{r-2} m \cdot \mathrm{C}_1 n \cdot \mathrm{C}_1 p + \cdots \\ &+ \mathrm{C}_{r-3} m \cdot \mathrm{C}_3 n + \cdots + \mathrm{C}_{r-3} m \cdot \mathrm{C}_2 n \cdot \mathrm{C}_1 p + \cdots \\ &\equiv \Sigma \mathrm{C}_r m \cdot \mathrm{C}_s n \cdot \mathrm{C}_t p \cdots,\end{aligned}$$

wherein $r, s, t, \cdots$ are any numbers such that $r+s+t+\cdots = m+n+p+\cdots$.

25. Of n things, all different, taken r at a time, repetitions allowed, there are n^r permutations.

26. Discuss the general case $\mathrm{P}_r n_{p \text{ alike}, q \text{ alike}, \ldots}$.

27. Discuss the general case $\mathrm{C}_r n_{p \text{ alike}, q \text{ alike}, \ldots}$.

V. POWERS AND ROOTS OF POLYNOMIALS.

§ 1. PRODUCT OF BINOMIAL FACTORS.

LEMMA. *If there be* n *binomial factors* $(x+a)$, $(x+b)$, $\cdots$ $(x+l)$, *their product is* $x^n+\Sigma_1(a\cdots l)\cdot x^{n-1}+\Sigma_2(a\cdots l)\cdot x^{n-2}+\cdots$ $+\Sigma_r(a\cdots l)\cdot x^{n-r}+\cdots+\Sigma_n(a\cdots l)$, *wherein* $\Sigma_1(a\cdots l)$, $\Sigma_2(a\cdots l)$, $\cdots\equiv$ *the sums of the products of the combinations of the terms* $a\cdots l$, *taken one at a time, taken two at a time, etc.* [IV. pr. 2 nt.

For $\because$ the product $(x+a)\cdot(x+b)\cdots(x+l)$ is the sum of the partial products of each term of the binomial factor $(x+a)$ by each term of the binomial factor $(x+b)$ by $\cdots$, [II. th. 5

$\therefore$ that product $=x^n$, the single product of the first terms of all the n binomials,

$+\Sigma_1(a\cdots l)\cdot x^{n-1}$, the sum of the n partial products formed, each of them, by taking the second term of one binomial, and multiplying by the first terms of all the other binomials,

$+\Sigma_2(a\cdots l)\cdot x^{n-2}$, the sum of the c_2n partial products formed by taking all possible combinations, two at a time, of the second terms of the several binomials, multiplying these two terms together, and multiplying each such product by the first terms of all the other binomials,

$+\cdots \quad \cdots \quad \cdots,$

$+\Sigma_r(a\cdots l)\cdot x^{n-r}$, the sum of the c_rn partial products formed by taking all possible combinations, r at a time, of the second terms of the several binomials, multiplying these terms together, and multiplying each such product by the first terms of all the other $n-1$ binomials,

$+\cdots \quad \cdots,$

$+a\cdot b\cdots l$, the single product of the last terms of all the n binomials. Q. E. D.

§ 2. THE BINOMIAL THEOREM.

THEOR. 1. *If a binomial be raised to any positive integral power, that power consists of the sum of a series, whose successive terms are the products of three factors:*

1. *The powers of the first term of the binomial, beginning with that power whose exponent is the exponent of the binomial, and decreasing by a unit for each term to the* 0*th power.*
2. *The powers of the second term of the binomial, beginning with the* 0*th power, and increasing by a unit for each term.*
3. *The number of combinations of a number of things equal to the number of units in the exponent, taken* 0, 1, 2, 3, ⋯ *at a time.*

Let $x+a$ be any binomial, and n any positive integer, then will

$$1]\quad (x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \frac{n(n-1)(n-2)}{3!}a^3x^{n-3}$$
$$+ \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^rx^{n-r}$$
$$+ \cdots + na^{n-1}x + a^n.$$

For, in the equation

$$(x+a)\cdot(x+b)\cdots(x+l)$$
$$= x^n + \Sigma_1(a\cdots l)\cdot x^{n-1} + \Sigma_2(a\cdots l)\cdot x^{n-2} + \cdots \quad \text{[lemma}$$

put a for each of the numbers $b, c, \cdots l$;

then $\because\ (x+a)\cdot(x+b)\cdots(x+l) = (x+a)\cdot(x+b)\cdots n$ factors $= (x+a)^n$;

and
$$\because\ \Sigma_1(a\cdots l) = a + a + a \cdots n \text{ terms} = \mathrm{c}_1 n\cdot a,$$
$$\Sigma_2(a\cdots l) = a^2 + a^2 + a^2 \cdots \mathrm{c}_2 n \text{ terms} = \mathrm{c}_2 n\cdot a^2,$$
$$\cdots \quad \cdots \quad \cdots$$
$$\Sigma_r(a\cdots l) = a^r + a^r + a^r \cdots \mathrm{c}_r n \text{ terms} = \mathrm{c}_r n\cdot a^r,$$
$$\cdots \quad \cdots \quad \cdots$$
$$\Sigma_{n-1}(a\cdots l) = a^{n-1} + a^{n-1} + \cdots \mathrm{c}_{n-1} n \text{ terms} = \mathrm{c}_{n-1} n\cdot a^{n-1},$$
$$\Sigma_n(a\cdots l) = a^n \text{ once} = \mathrm{c}_n n\cdot a^n;$$
$$\therefore\ (x+a)^n = \mathrm{c}_0 n\cdot a^0\cdot x^n + \mathrm{c}_1 n\cdot a\cdot x^{n-1} + \mathrm{c}_2 n\cdot a^2\cdot x^{n-2}$$
$$+ \cdots + \mathrm{c}_r n\cdot a^r\cdot x^{n-r} + \cdots + \mathrm{c}_{n-1} n\cdot a^{n-1}\cdot x + \mathrm{c}_n n\cdot a^r\cdot x^0,$$

i.e.,
$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} + \cdots$$
$$+ \frac{n(n-1)\cdots(n-r+1)}{r!}a^rx^{n-r} + \cdots + na^{n-1}x + a^n.$$

Q. E. D.

NOTE 1. The theorem is also proved by aid of [II. th. 5].

For $\quad (x+a)^n = (x+a)\cdot(x+a)\cdot(x+a)\cdots n$ factors

$$\begin{array}{r|l|l}
= x\cdot x\cdot x\cdots x + x\cdot x\cdot x\cdots a & + x\cdot x\cdot x\cdots a\cdot a & + \cdots + a\cdot a\cdot a\cdots a \\
+ \cdots\ \ \cdots\ \ \cdots & + \cdots\ \ \ \cdots\ \ \ \cdots & \\
+ x\cdot x\cdot a\cdots x & + x\cdot x\cdot a\cdots x\cdot a & \\
+ x\cdot a\cdot x\cdots x & + x\cdot a\cdot x\cdots x\cdot a & \\
+ a\cdot x\cdot x\cdots x & + \cdots\ \ \ \cdots\ \ \ \cdots & \\
 & + a\cdot a\cdot x\cdots x\cdot x &
\end{array}$$

$$= \mathrm{P}_n n_{\,n\text{ alike}}\cdot x^n + \mathrm{P}_n n_{\,n-1\text{ alike}}\cdot a\cdot x^{n-1} + \mathrm{P}_n n_{\,n-2\text{ alike},\,2\text{ alike}}\cdot a^2\cdot x^{n-2}$$

$$+ \cdots + \mathrm{P}_n n_{\,n-r\text{ alike},\,r\text{ alike}}\cdot a^r x^{n-r} + \cdots + \mathrm{P}_n n_{\,n\text{ alike}}\cdot a^n$$

$$= x^n + nax^{n-1} + \frac{n(n-1)}{2\,!}a^2x^{n-2} + \cdots + a^n. \quad \text{Q. E. D. [IV. th. 2}$$

NOTE 2. The theorem is also proved by induction.

1. *The law is true for the second power.*

For $\because\ (x+a)^2 = x^2 + 2ax + a^2,$ [multiplication

$\therefore\ (x+a)^n = x^n + nax^{n-1} + \cdots + a^n,$ when $n = 2$.

2. *If the law be true for the* k*th power, it is also true for the* (k+1)*th.*

For, write $(x+a)^k = x^k + kax^{k-1} + \dfrac{k(k-1)}{2\,!}a^2x^{k-2} + \cdots$

$$+ \frac{k(k-1)\cdots(k-h+1)}{h\,!}a^hx^{k-h} + \cdots + a^k. \quad \text{[hyp.}$$

Multiply both members by $x+a$;

then
$$(x+a)^{k+1} = x^{k+1} + \begin{array}{r|l|l} k & ax^k + \dfrac{k(k-1)}{2\,!} & a^2x^{k-1} + \cdots \\ +1 & \quad +k & \end{array}$$

$$+ \begin{array}{l|l} \dfrac{k(k-1)\cdots(k-h+1)}{h\,!} & a^hx^{k-h+1} + \cdots + a^{k+1} \\ + \dfrac{k(k-1)\cdots(k-h+2)}{(h-1)\,!} & \end{array}$$

$$= x^{k+1} + (k+1)ax^k + \frac{(k+1)k}{2\,!}a^2x^{k-1} + \cdots$$

$$+ \frac{(k+1)\,k(k-1)\cdots(k-h+2)}{h\,!}a^hx^{k-h+1} + \cdots + a^{k+1}.$$

3. *The law is true, whatever the exponent* k.

For $\because$ it is true for $k = 2$, [1

$\therefore$ it is true for $k = 3$. [2

So for $k = 4$, for $k = 5$, $\cdots$ for $k = n$. Q. E. D.

Cor. 1. *If* x *and* a *be any numbers and* n *any positive integer,*

2] $$(x-a)^n = x^n - nax^{n-1} + \frac{n(n-1)}{2!}a^2x^{n-2} - \cdots \mp nax^{n-1} \pm a^n.$$

Cor. 2. *The series is finite.*

For $\because$ the series is a continued product of finite polynomials,

$\therefore$ it is itself finite. Q.E.D.

Note. Another and independent proof of Cor. 2 is as follows:

For $\because$ the several coefficients form a series

$$1,\ \frac{n}{1},\ \frac{n}{1}\cdot\frac{n-1}{2},\ \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3},\ \cdots,$$

wherein each term is formed by multiplying the preceding one by

a fraction of the form $\frac{n-k}{k+1}$; [IV. th. 3 nt.

and $\because$ the numerator of this fraction grows less by a unit at each step, and the denominator greater,

$\therefore$ some term of the series, and all after it, is 0, and the series terminates.

Cor. 3. *The coefficients of any two terms equally distant from the extremities of the development are identical.*

Cor. 4. *The sum of the coefficients of* $(x+a)^n$ *is* 2^n.

For, let $x = 1$, $a = 1$;

then $\because$ $(x+a)^n = (1+1)^n = 2^n$,

and $\because$ $(1+1)^n = 1^n + n\cdot 1\cdot 1^{n-1} + \frac{n(n-1)}{2!}\cdot 1^2\cdot 1^{n-2} + \cdots + 1^n$ [th.

$$= 1 + n + \frac{n(n-1)}{2!} + \cdots + 1,$$

$\therefore$ $2^n = 1 + n + \frac{n(n-1)}{2!} + \cdots + 1.$ Q.E.D.

Cor. 5. *The sum of the coefficients of* $(x-a)^n$ *is* 0.

Cor. 6. *In the development of* $(x+a)^n$ *the sum of* 1*st*, 3*d*, 5*th*, $\cdots$ *coefficients, and the sum of the* 2*d*, 4*th*, 6*th*, $\cdots$ *coefficients, are equal; and each sum is* 2^{n-1}.

For $\because$ the sum of all of them is 2^n, [cr. 4

and $\because$ sum (1st + 3d + $\cdots$) − sum (2d + 4th + $\cdots$) = 0, [cr. 5

$\therefore$ sum (1st + 3d + $\cdots$) = sum (2d + 4th + $\cdots$) = $2^n : 2 = 2^{n-1}$.

NOTE. Cors. 4, 5, 6 may be written in formula, thus:

3] $\quad c_n n + c_{n-1} n + c_{n-2} n + \cdots + c_2 n + c_1 n = 2^n,$

4] $\quad c_n n - c_{n-1} n + c_{n-2} n - \cdots \mp c_2 n \pm c_1 n = 0,$

5] $\quad c_n n + c_{n-2} n + c_{n-4} n + \cdots = c_{n-1} n + c_{n-3} n + c_{n-5} n = 2^{n-1}.$

§ 3. THE POLYNOMIAL THEOREM.

THEOR. 2. *If* a, b, c, ⋯ l *be any* m *numbers;* n *a positive integer;* p, q, r, ⋯ z *any positive integers* (*including* 0), *such that* p + q + r + ⋯ + z = n, *then:*

6] $$(a+b+c+\cdots+l)^n$$
$$\begin{aligned}
&= \frac{n!}{n!\cdot 0!\cdot 0!\cdots}\cdot \Sigma a^n \cdot b^0 \cdot c^0 \cdots l^0 \\
&+ \frac{n!}{(n-1)!\cdot 1!\cdot 0!\cdots}\cdot \Sigma a^{n-1} \cdot b^1 \cdot c^0 \cdots l^0 \\
&+ \frac{n!}{(n-2)!\cdot 2!\cdot 0!\cdots}\cdot \Sigma a^{n-2} \cdot b^2 \cdot c^0 \cdots l^0 \\
&+ \frac{n!}{(n-2)!\,.\,1!\,.\,1!\,.\,0!\cdots}\cdot \Sigma a^{n-2} \cdot b^1 \cdot c^1 \cdot d^0 \cdots l^0 \\
&+ \frac{n!}{(n-3)!\cdot 3!\cdot 0!\cdots}\cdot \Sigma a^{n-3} \cdot b^3 \cdot c^0 \cdots l^0 \\
&+ \frac{n!}{(n-3)!\cdot 2!\cdot 1!\cdot 0!\cdots}\cdot \Sigma a^{n-3} \cdot b^2 \cdot c^1 \cdot d^0 \cdots l^0 \\
&+ \frac{n!}{(n-3)!\cdot 1!\cdot 1!\cdot 1!\cdot 0!\cdots}\cdot \Sigma a^{n-3} \cdot b^1 \cdot c^1 \cdot d^1 \cdot e^0 \cdots l^0 \\
&+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
&+ \frac{n!}{p!\cdot q!\cdot r!\cdots z!}\cdot \Sigma a^p \cdot b^q \cdot c^r \cdots l^z \quad \text{[the general term} \\
&+ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots.
\end{aligned}$$

This theorem is but the generalization of the binomial theorem, and is proved in the same way.

The reader may review here what is said of symmetry in multiplication [II. pr. 3, nt. 7]. He may also compare [IV. th. 5]. He will observe that he is actually forming the homogeneous products there spoken of. They are, however, of the nth degree here, instead of the kth degree as there, and there are m numbers instead of n.

COR. 1. *Let* $a + bx + cx^2 + dx^3 + \cdots$ *be a series arranged to ascending powers of* x; *then will*

7] $$(a + bx + cx^2 + dx^3 + \cdots)^n$$
$$= \frac{n!\cdot a^n}{n!} + \frac{n!\cdot a^{n-1}b}{(n-1)!\,1!}\Bigg|\, x + \begin{array}{l} \dfrac{n!\cdot a^{n-2}b^2}{(n-2)!\,2!} \\ + \dfrac{n!\cdot a^{n-1}c}{(n-1)!\,1!} \end{array}\Bigg|\, x^2 + \begin{array}{l} \dfrac{n!\cdot a^{n-3}b^3}{(n-3)!\,3!} \\ + \dfrac{n!\cdot a^{n-2}bc}{(n-2)!\,1!\,1!} \\ + \dfrac{n!\cdot a^{n-1}d}{(n-1)!\,1!} \end{array}\Bigg|\, x^3 + \cdots$$
$$= a^n + na^{n-1}bx + \begin{array}{l} \dfrac{n(n-1)}{2!}a^{n-2}b^2 \\ + na^{n-1}c \end{array}\Bigg|\, x^2 + \begin{array}{l} \dfrac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \\ + n(n-1)a^{n-2}bc \\ + na^{n-1}d \end{array}\Bigg|\, x^3 + \cdots;$$

and, if p *be any positive integer, and* $r, s, t \cdots$ *be any other positive integers, such that* $0\cdot r + 1\cdot s + 2\cdot t + \cdots = p$, *the coefficient of* x^p *in the development is* $\Sigma \dfrac{n!}{r!\cdot s!\cdot t!\cdots}\cdot a^r\cdot b^s\cdot c^t \cdots$.

COR. 2. *If all the* m *numbers* $a, b, c, \cdots l$ *be positive, the sum of the coefficients of the development of* $(a + b + c + \cdots l)^n$ *is* m^n; *if one of them be negative, the sum is* $(m - 2)^n$; *if two of them be negative, the sum is* $(m - 4)^n, \cdots$; *and so on.*

COR. 3. *The development has as many sums of symmetric terms of the form given above as there are ways in which* m *positive integers* $p, q, r, \cdots z$ *can be chosen, so that their sum shall be* n.

E.g., if $m = 4$, and $n = 6$, the four integers p, q, r, s may be either of the following:

6, 0, 0, 0;	5, 1, 0, 0;	4, 2, 0, 0;
4, 1, 1, 0;	3, 3, 0, 0;	3, 2, 1, 0;
3, 1, 1, 1;	2, 2, 2, 0;	2, 2, 1, 1;

and there are nine terms in the development.

COR. 4. *The development has* $\dfrac{(n + m - 1)!}{n!\cdot(m - 1)!}$ *separate terms.*

For this is the greatest number of terms possible in any integral polynomial of the nth degree homogeneous and having m letters.

E.g., $(a+bx+cx^2)^4$ has $\dfrac{6!}{4!\cdot 2!}$, $=15$, separate terms. [IV. th. 6

§ 4. ROOTS OF POLYNOMIALS.

PROB. 1. TO FIND THE nth ROOT OF A POLYNOMIAL.

Arrange the terms of the polynomial in the order of the powers of some one letter, a perfect power first.

If the first term be not a perfect power, divide the polynomial by such a monomial as will make it a perfect power, and reserve the root of this monomial as a factor of the result.

Take the n*th root of the first term.*

Raise this root to the (n−1)*th power and multiply by* n.

Divide the second term of the polynomial by this product (the trial divisor) *and add the quotient to the root first found.*

Raise the whole root to the n*th power and subtract it from the polynomial.*

Divide the first term of the remainder by the trial divisor; add the quotient to the root found; raise the whole root to the n*th power; subtract from the polynomial; and so on.*

Let $\text{P} \equiv$ the given polynomial, and $\text{A} + \cdots \text{D} + \text{E} + \cdots \equiv$ its nth root, both arranged by $\left\{\begin{matrix}\text{descending}\\ \text{ascending}\end{matrix}\right.$ powers of some letter x; and let $\text{A} + \cdots \text{D} \equiv$ the terms already found;

then $\because\ \text{P} - \overline{\text{A} + \cdots \text{D}}^n = (\text{A} + \cdots \text{D} + \text{E} + \cdots)^n - \overline{\text{A} + \cdots \text{D}}^n$

$= n\text{A}^{n-1} \cdot \text{E} +$ terms with $\left\{\begin{matrix}\text{lower}\\ \text{higher}\end{matrix}\right.$ powers of x,

$\therefore\ \text{E} \equiv$ first term of quotient, $(\text{P} - \overline{\text{A} + \cdots \text{D}}^n) : n\text{A}^{n-1}$,

and $\text{P} - \overline{\text{A} + \cdots \text{D} + \text{E}}^n$,

$= n\text{A}^{n-1} \cdot \text{F} +$ terms with $\left\{\begin{matrix}\text{lower}\\ \text{higher}\end{matrix}\right.$ powers of x,

has not the $\left\{\begin{matrix}\text{highest}\\ \text{lowest}\end{matrix}\right.$ power of x in $\text{P} - \overline{\text{A} + \cdots \text{D}}^n$.

So the successive terms of P are exhausted, as new terms of the root are found. Q. E. D.

NOTE 1. The work is an effort to retrace the steps taken in getting the power whose root is now sought. It is a process of trial, by progressive steps, like division and other inverse operations, and its success is established by raising the root to the required power and comparing it with the given polynomial.

[II. § 2, p. 29

NOTE 2. COMPLETE DIVISOR: In square root and cube root certain modifications may be introduced into the rule which shorten the work:

In square root the trial divisor is double the first term of the root; and a *complete divisor* is got by doubling the root already found and adding the new term of the root. When the complete divisor is multiplied by this new term of the root, and the product is subtracted from the last remainder, the whole root found is thereby squared and subtracted from the polynomial.

E.g.,

$$\begin{array}{lll} & a^2+2ab+b^2+2ac+2bc+c^2 & \underline{|\,a+b+c} \\ & a^2 & \\ \hline 2a+b\,] & 2ab+b^2 & \\ & 2a+2b+c\,] & 2ac+2bc+c^2 \end{array}$$

In cube root the trial divisor is three times the square of the first term of the root, and the complete divisor is the sum of three times the square of the root already found, three times the product of this root by the new term of the root, and the square of the new term; and when the complete divisor is multiplied by the new term and subtracted from the last remainder, the whole root found is thereby cubed and subtracted from the polynomial.

E.g., $\underline{|\,a+b+c}$

$$\begin{array}{ll} & a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3ac^2+3b^2c+3bc^2+c^3 \\ & a^3 \\ \hline 3a^2+3ab+b^2\,] & 3a^2b+3ab^2+b^3 \\ 3a^2+6ab+3b^2+3ac+3bc+c^2\,] & 3a^2c+6abc+3ac^2+3b^2c+3bc^2+c^3 \end{array}$$

The reader may deduce like rules for getting the 4th, 5th, ⋯ roots, by means of the complete divisor, from the formula

$$A^n+\left(nA^{n-1}+\frac{n(n-1)}{2!}A^{n-2}B+\cdots+B^{n-1}\right)\cdot B=(A+B)^n, \quad [1$$

NOTE 3. ROOTS OF ROOTS: For a root whose index is composite, it is generally better to factor the index and take in succession the roots indicated by such factors. [II. th. 3 cr. 9, nt.

E.g., the 4th root is the square root of the square root;
the 6th root is the cube root of the square root;
the 8th root is the square root of the square root of the square root; and so on.

NOTE 4. ROOTS OF FRACTIONS: To find the root of a fraction, write the root of the numerator over that of the denominator.

§ 5. ABSOLUTE AND RELATIVE ERROR.

When a number is given approximately only, the *absolute error* is the excess of the assumed value above the true value; and it is $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ if the assumed value be $\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right.$ than the true value. The *relative error* is the ratio, absolute error : true value.

E.g., if of a long decimal a few of the figures be taken and the rest dropped, the absolute error is smaller than a unit of the order of the last figure taken; but if the last figure taken be made one greater when the first figure dropped is five or more, then the absolute error is not larger than a half unit of the order of the last figure taken.

A number is $\left\{\begin{matrix}\textit{correct}\\ \textit{approximate}\end{matrix}\right.$ *to* n *figures* when its absolute error is not larger than $\left\{\begin{matrix}\frac{1}{2}\\ 1\end{matrix}\right.$ unit in its nth place towards the right.

E.g., if $x \sim .2037 \not> .0005$, then .204 is approximate to three figures, and .20 is correct to two figures.

So, for $100x$, 20.4 is approximate to three figures.

The copula $\doteq$, read *approaches*, joins numbers which differ by a number very small as to either of them. It is, therefore, used to join an assumed value to the true value of a number when the relative error becomes very small.

E.g., if a be the true value of a number, x the assumed value, and α the absolute error, then $x = a + \alpha$, and $x \doteq a$ when α becomes very small.

So, $3a + \alpha^2 \doteq 3a$ when α becomes very small.

In numerical work the *degree of approximation* depends on the relative error.

E.g., an inch in the earth's diameter, and a million miles in a star's distance are alike inappreciable; but a thousandth of an inch in a microscopic measurement is enormous.

In pure mathematics the degree of approximation depends solely upon the time and patience of the computer; but of numbers based on measurement the possible relative error is seldom smaller than a millionth.

THEOR. 3. *If a number be approximate to* n *significant figures and no more, the possible relative error* $> 1:10^n$ *and* $\ngtr 1:10^{n-1}$.

For $\because$ any number $\leqslant 10^n$ units of its own nth place,

and $\nless 10^{n-1}$ such units,

and $\because$ poss. abs. err. $\equiv 1$ such unit, [hyp.

and $\because$ poss. rel. err. $=$ poss. abs. err. : true number, [df.

$\therefore$ poss. rel. err. $> 1:10^n$ and $\ngtr 1:10^{n-1}$. Q. E. D. [II. ax. 18

COR. *A number whose relative error is not larger than* $1:10^n$ *is approximate to at least* n *significant figures.*

For $\because$ the number $\leqslant 10^n$ units of the nth place,

and $\because$ its rel. err. $\ngtr 1:10^n$, [hyp.

$\therefore$ its abs. err., $=$ number $\times$ rel. err.,

$\leqslant 1$ unit of the nth place;

i.e., the number is approximate to n figures. Q. E. D.

THEOR. 4. *The* $\left\{\begin{matrix}\textit{absolute}\\ \textit{relative}\end{matrix}\right.$ *error of the* $\left\{\begin{matrix}\textit{sum}\\ \textit{product}\end{matrix}\right.$ *of two or more numbers* $\left\{\begin{matrix}\textit{equals}\\ \textit{approaches}\end{matrix}\right.$ *the sum of their* $\left\{\begin{matrix}\textit{absolute}\\ \textit{relative}\end{matrix}\right.$ *errors.*

For, let $a, b, \cdots \equiv$ the true values of two or more numbers, $x, y, \cdots \equiv$ their assumed values, $\alpha, \beta, \cdots \equiv$ their absolute errors; then:

(a) $\because x+y+\cdots = (a+\alpha)+(b+\beta)+\cdots$

$= (a+b+\cdots)+(\alpha+\beta+\cdots)$,

$\therefore$ the abs. err. of sum, $x+y+\cdots$, $=\alpha+\beta+\cdots$,

i.e., $=$ sum of abs. errs. Q. E. D.

(b) $\because x\cdot y\cdots = (a+\alpha)\cdot(b+\beta)\cdots$

$= a\cdot b\cdots +$ terms which contain either α and not a, or β and not b, or $\cdots$, as a factor, $+$ terms with two or more of the abs. errs. $\alpha, \beta, \cdots$ as factors,

$\therefore$ abs. err. prod. $x\cdot y\cdots$, $\equiv x\cdot y\cdots - a\cdot b\cdots$,

$=$ the sum of terms all having one or more of the abs. errs. $\alpha, \beta, \cdots$ for factors;

$\therefore$ rel. err. prod. $x\cdot y\cdots = \dfrac{\alpha}{a}+\dfrac{\beta}{b}+\cdots+\dfrac{\alpha\cdot\beta}{a\cdot b}+\cdots$;

i.e., rel. err. prod. $x\cdot y\cdots \doteq \dfrac{\alpha}{a}+\dfrac{\beta}{b}+\cdots =$ sum of rel. errs. Q. E. D.

COR. 1. *If the abs. errs.* $\alpha, \beta, \cdots$ *be each not larger than* ϵ, *and if* m, n, $\cdots$ *be any finite numbers, then*
abs. err. $(\mathrm{mx} + \mathrm{ny} + \cdots) = \mathrm{m}\alpha + \mathrm{n}\beta + \cdots \not> (^+\mathrm{m} + {}^+\mathrm{n} + \cdots) \cdot \epsilon$.

In particular: abs. err. $(x - y) = \alpha - \beta, \not> 2\,\epsilon$.

COR. 2. *Rel. err.* mx = *rel. err.* x, *if* m *have no error.*

COR. 3. *The relative error of a quotient approaches the difference of the relative errors of the elements.*

For $\because$ divd. = divr. × quot.,

$\therefore$ rel. err. divd. $\doteq$ rel. err. divr. + rel. err. quot.; [th.

$\therefore$ rel. err. quot. $\doteq$ rel. err. divd. — rel. err. divr. Q. E. D.

In particular: the relative error of the reciprocal of a number approaches the opposite of the relative error of the number itself.

THEOR. 5. *The relative error of a positive integral power of a number approaches the product of the relative error of the number by the exponent of the power; and that of a root approaches the quotient of the relative error of the number by the root-index.*

Let x be any approximate number, and n any positive integer; then will rel. err. $x^n \doteq n \cdot$ rel. err. x, and rel. err. $\sqrt[n]{x} \doteq \frac{1}{n} \cdot$ rel. err. x.

(a) $\because$ $x^n, \equiv (a+\alpha)^n, = a^n + n \cdot a^{n-1} \cdot \alpha + \frac{n \cdot (n-1)}{2\,!} \cdot a^{n-2} \cdot \alpha^2 + \cdots,$

$\therefore$ rel. err. $x^n, \equiv \frac{x^n - a^n}{a^n}, = n \cdot \frac{\alpha}{a} + \frac{n \cdot (n-1)}{2\,!} \cdot \left(\frac{\alpha}{a}\right)^2 + \cdots,$

$\doteq n \cdot \frac{\alpha}{a} = n \cdot$ rel. err. x. Q. E. D.

(b) $\because$ $x = (\sqrt[n]{x})^n$, [I. § 11, df.

and $\because$ rel. err. x, $\equiv$ rel. err. $(\sqrt[n]{x})^n$, $\doteq n \cdot$ rel. err. $\sqrt[n]{x}$; [(a)

$\therefore$ rel. err. $\sqrt[n]{x} \doteq \frac{1}{n} \cdot$ rel. err. x. Q. E. D.

NOTE. Ths. 3–5 enable the computer: (a) to see how far his results can be depended upon as approximate; (b) to carry each part of his work so far that the final result shall be as approximate as he desires, or as the data, if themselves only approximate, permit, wasting no labor upon needless or unreliable figures. Results correct to the last figure, *e.g.* for standard tables, are only got by computing with several extra decimals.

§ 6. ROOTS OF NUMERALS.

PROB. 2. TO FIND THE nTH ROOT OF A NUMERAL.

Separate the numeral into periods of n *figures each, both to the left and to the right of the decimal point.*

Take the n*th root of the largest perfect* n*th power contained in the left-hand period.*

Subtract this power from the period, and to the remainder annex the next period to form the first dividend.

Raise the root-figure to the (n−1)*th power, and multiply by* n.

Divide the first dividend by this product (the trial divisor), *and annex the quotient-figure to the root first found.*

Raise the whole root to the n*th power, subtract from the first two periods, and to the remainder annex the next period for the second dividend.*

Raise the root found to the (n−1)*th power, and multiply by* n *for a new trial divisor; and so on.*

NOTE 1. Numerals are polynomials, but polynomials in which the terms overlie and hide each other; and virtually the rule is the same for finding the roots of both.

The separation into periods is a matter of convenience only. It comes from this: that the figures of the root of different orders are best got separately, and that, since the nth power of even tens has n 0's, therefore the first n figures, counting from the decimal point to the left, are of no avail in getting the tens of the root, and are set aside and reserved till wanted in getting the units' figure. So the nth power of even hundreds has $2n$ 0's, and the first $2n$ figures, two periods, are set aside and reserved till wanted in getting the tens; and so on.

So, in getting roots of decimal fractions, the nth power of tenths has n decimal figures, and the first n figures, one period, are used in getting the tenths' figure of the root; the nth power of hundredths has $2n$ decimal figures, and so on. The same thing appears from this: that the root is easiest got if the denominator be a perfect nth power; and this it is only when it consists of 1 with n 0's, or $2n$ 0's, or $3n$ 0's, and so on; that is, when the number of decimal figures used is n, or $2n$, or $3n$, and so on.

NOTE 2. APPROXIMATION: The root of a numeral may be got to any degree of approximation by reducing it to a fraction whose denominator equals or exceeds the nth power of the denominator sought, and then extracting the root.

E.g., $\sqrt[3]{4}$ to $\frac{1}{12}$th: $4=\frac{6912}{1728}$ and $\sqrt[3]{\frac{6912}{1728}}=\frac{19+}{12}$.

NOTE 3. SQUARE ROOT BY CONTRACTION: When the first n figures of the root of a numeral have been got by the rule above, then $n-1$ more figures may be got by dividing the remainder by double this root.

For $\because$ rel. err. quotient $\doteq$ opp. rel. err. divisor, [th. 4 cr. 2
i.e., $\doteq$ opp. rel. err. $2\cdot$ first n figures,
$\doteq$ opp. rel. err. first n figures [th. 4 cr. 2
$\not> 1:10^{n-1}$, [th. 3
$\therefore$ quotient is approx. to $(n-1)$ figures. Q.E.D. [th. 3 cr.

NOTE 4. CUBE ROOT BY CONTRACTION: When the first n figures of a root have been got by the rule above, then $n-2$ more figures may be got by dividing the remainder by three times the square of this root.

For $\because$ rel. err. quotient $\doteq$ opp. rel. err. divisor,
i.e., $\doteq$ opp. rel. err. $3\cdot(\text{first } n \text{ figures})^2$,
$\doteq$ opp. $2\cdot$ rel. err. first n figures
[th. 5, th. 4 cr. 1
$\not> 2:10^{n-1} \lessdot 1:10^{n-2}$,
$\therefore$ quotient is approx. to at least $(n-2)$ figures. Q.E.D.

§ 7. EXAMPLES.

§ 2. THEOR. 1.

1. Expand $(1+x)^5$, $(a+b)^7$, $(3a-2x^2)^5$, $(a+bx+cx^2)^4$.

NOTE. To expand a trinomial, bracket two terms and apply the formula both to the whole expression and to the powers of the bracket, thus $(a+\overline{bx+cx^2})^4$.

2. In $(x+y)^{10}$, show that the sum of the coefficients of the odd terms equals the sum of the coefficients of the even terms.

3. Write down that term of the expansion of $\left(x+\frac{1}{x}\right)^n$ which does not contain x when n is even.

4. Write down the 8th term, and the largest term, of $\left(1+\frac{1}{3}\right)^{18}$.

NOTE. To determine the largest term, observe the factors by which the successive terms are multiplied to get the next terms in order. These multipliers constantly grow smaller; and when first one of them falls below a unit, then the term last before it is the largest, and those which follow are successively smaller and smaller. Sometimes two successive terms, equally large, are larger than any of the others.

5. By means of the binomial theorem show that the number of all possible combinations of 8 things is 255.

6. Show that the coefficient of the 9th term in the expansion of $(1+x)^{12}$ is equal to the sum of the coefficients of the 8th and the 9th term of the expansion of $(1+x)^{11}$.

7. In Ex. 6 write n in place of 11 and r in place of 8, and make the proof general.

8. Show that the middle term of the expansion of $(1+x)^{2n}$ is $1\cdot3\cdot5\cdots(2n-1)\cdot2^n\cdot x^n : n!$

9. If the coefficients of the $(r+1)$th and the $(r+3)$th terms of the expansion of $(1+x)^n$ be equal, find r.

10. If $\text{N} \equiv$ the nth term of the expansion of $(1-x)^m$, then the series, after the first n terms, is

$$\text{N}\cdot x\cdot\left(1-\frac{m+1}{n}\right)+\text{N}\cdot x^2\cdot\left(1-\frac{m+1}{n}\right)\cdot\left(1-\frac{m+1}{n+1}\right)+\cdots.$$

§ 3. THEOR. 2.

11. Write seven terms of the expansion of $(a+bx+cx^2+\cdots)^4$.

12. Write the expansion of $(1-5x+3x^2)^3$.

13. Write eight terms of the expansion, $(1-x-x^2+x^3+x^4-\cdots)^2$.

14. Write the expansion of $(\mathrm{A}+\mathrm{B}x^2+\mathrm{C}x^4+\cdots)^5$ as far as x^{10}.

15. Expand $(a+b+c+d)^4$, $(a+b+c+d+e)^5$, in sums of symmetric terms. [6

How many unlike terms in each of these sums?

How many partial products in each sum?

Check the work by showing that the number of unlike terms in all the sums is as in [IV. th. 6], and that the number of partial products in all the sums is as in [II. th. 5 cr. 7].

§ 4. PROB. 1.

16. Extract the square roots of:

$16x^2-40xy+25y^2$; $1+2x+7x^2+6x^3+9x^4$;

$9x^2-30ax-3a^2x+25a^2+5a^3+\dfrac{a^4}{4}$; $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}-\dfrac{x}{y}+\dfrac{y}{x}-\dfrac{7}{4}$.

$1+x^2$, x^2+1, x^2-a^2, and a^2-x^2, each to 4 terms.

17. Extract the cube roots of:

$1+6x+12x^2+8x^3$; $a^6-9a^4b^3c+27a^2b^6c^2-27b^9c^3$.

$1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$;

$\dfrac{x^6}{y^3}-6x^4+12x^2y^3-8y^6$; $\dfrac{x^3}{8}+\dfrac{1}{2}+\dfrac{2}{3x^3}+\dfrac{8}{27x^6}$;

$(a+1)^{6n}x^3-6ca^p(a+1)^{4n}x^2+12c^2a^{2p}(a+1)^{2n}x-8c^3a^{3p}$.

18. Extract the 4th root of

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4+4x^3z+12x^2yz+12xy^2z$$
$$+4y^3z+6x^2z^2+12xyz^2+6y^2z^2+4xz^3+4yz^3+z^4,$$

i.e., of $\Sigma x^4+\Sigma 4x^3y+\Sigma 6x^2y^2+\Sigma 12x^2yz$.

19. Extract the 5th root of

$$x^{10}-\frac{5}{2}x^8y+\frac{5}{2}x^6y^2-\frac{5}{4}x^4y^3+\frac{5}{16}x^2y^4-\frac{1}{32}y^5.$$

20. Extract the 6th root of [at one operation, or at two

$$a^6-2a^5b+\frac{5}{3}a^4b^2-\frac{20}{27}a^3b^3+\frac{5}{27}a^2b^4-\frac{2}{81}ab^5+\frac{1}{729}b^6.$$

§ 6. PROB. 2.

21. Extract the square root, each to three decimal places, of
144, 14.4, 1.44, .144, .0144, .00144, .000144.

22. Extract the cube root, each to three decimal places, of
1728, 172.8, 17.28, 1.728, .1728, .01728, .001728.

23. Find the values, each correct to within $\frac{1}{8}$, of
$$\sqrt{17},\ \sqrt{\frac{17}{9}},\ \sqrt{\frac{17}{27}},\ \sqrt{\frac{17}{81}},\ \sqrt{\frac{81}{12}},\ \sqrt{\frac{81}{36}},\ \sqrt{\frac{81}{63}},\ \sqrt{\frac{81}{324}}.$$

24. Find the values, each correct to within $\frac{1}{6}$, of
$$\sqrt[3]{53},\ \sqrt[3]{\frac{53}{12}},\ \sqrt[3]{\frac{53}{9}},\ \sqrt[3]{\frac{53}{27}},\ \sqrt[4]{\frac{53}{6}},\ \sqrt[5]{\frac{53}{36}},\ \sqrt[6]{\frac{53}{216}},\ \sqrt[7]{\frac{53}{729}}.$$

25. Find the values, each to five decimal places, of [contraction
$$\sqrt{185},\ \sqrt{912},\ \sqrt[3]{625},\ \sqrt[3]{587},\ \sqrt[4]{729},\ \sqrt[4]{1008},\ \sqrt[6]{1728}.$$

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26. If $x^{n,d} \equiv x \cdot (x+d) \cdot (x+2d) \cdots (x+\overline{n-1} \cdot d)$ [n a pos. integer
show that $x^{n,0} = x^n$, $x^{1,d} = x$, $x^{0,d} = 1$.

27. If $P \equiv$ any homogeneous polynomial of the nth degree as to $x, y, \cdots$, and if $Q, R \equiv$ what P, $(x+y+\cdots) \cdot P$ become when for $x^2, x^3, \cdots y^2, \cdots$ are put $x^{2,d}, x^{3,d}, \cdots y^{2,d}, \cdots$,
show that every term $ax^{r,d}y^{s,d}\cdots$ of Q gives in the product $(n+x+y+\cdots) \cdot Q$ the terms $ax^{r+1,d}y^{s,d}\cdots$, $ax^{r,d}y^{s+1,d}\cdots$, etc.; and hence, that $(n+x+y+\cdots) \cdot Q = R$.

28. Use the result of Ex. 27 to show that:
$$(x+y)^{n,d} = x^{n,d} + n \cdot x^{n-1,d} \cdot y + \frac{n(n-1)}{2!} \cdot x^{n-2,d} \cdot y^{2,d} + \cdots + y^{n,d},$$
$$(x+y+z+\cdots)^{n,d} = \Sigma \frac{n!}{p!\,q!\,r!\cdots} x^{p,d} y^{q,d} z^{r,d} \cdots.$$
[$p+q+r+\cdots = n$.

VI. CONTINUED FRACTIONS.

§ 1. FORM OF CONTINUED FRACTIONS. — CONVERGENTS.

In place of fractions with large terms, or of incommensurables, it is often convenient to use fractions with small terms, whose values are nearly equal to the true values of the given numbers. Such approximate fractions, when arranged in a series approaching more and more closely to the true value of a number, are called its *convergents*. The excess of a convergent over the true value is its *error*. Convergents are found in various ways; among others, by aid of continued fractions.

A *continued fraction* is an expression of the form

$$\cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cfrac{n_3}{d_3 + \ddots + \cfrac{n_k}{d_k + \cdots,}}}}$$

i.e., a complex fraction whose numerator is an entire number, and whose denominator is an entire number plus a fraction whose numerator is an entire number and whose denominator is an entire number plus, and so on.

The fractions $\frac{n_1}{d_1}, \frac{n_2}{d_2}, \frac{n_3}{d_3}, \cdots \frac{n_k}{d_k}$ are the 1st, 2d, 3d, $\cdots$ kth *partial fractions*, and $\left\{ \begin{matrix} n_1, n_2, n_3, \cdots n_k \\ d_1, d_2, d_3, \cdots d_k \end{matrix} \right.$ are the 1st, 2d, 3d, $\cdots$ kth *partial* $\left\{ \begin{matrix} \textit{numerators.} \\ \textit{denominators.} \end{matrix} \right.$ These partial numerators and denominators are here assumed to be entire numbers, and they may be either positive or negative.

The expressions

$$\frac{n_1}{d_1};\ \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2}}, = \frac{n_1 d_2}{d_1 d_2 + n_2};\ \cdots \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \ddots + \cfrac{n_k}{d_k}}}, = \cdots;$$

are called the 1st, 2d, $\cdots$ kth convergents, because, usually, they are true convergents; but sometimes they do not converge toward the true value, but diverge from it.

A continued fraction is $\left\{\begin{matrix}\textit{finite}\\ \textit{infinite}\end{matrix}\right.$ when the number of partial fractions, and therefore of convergents, is $\left\{\begin{matrix}\text{limited.}\\ \text{unlimited.}\end{matrix}\right.$

An infinite continued fraction $\left\{\begin{matrix}\textit{converges}\\ \textit{diverges}\end{matrix}\right.$ when its convergents, if carried far enough, $\left\{\begin{matrix}\text{will}\\ \text{will not}\end{matrix}\right.$ differ from the true value by less than any assigned number, however small.

§ 2. CONVERSION OF COMMON FRACTIONS.

PROB. 1. TO CONVERT A COMMON FRACTION INTO A CONTINUED FRACTION.

(a) *Numerical*, $n_1, n_2, \cdots$ *each* $= 1$:

If an improper fraction, reduce to a mixed number; then divide both terms of the fractional part by its numerator, both terms of the fractional part of the new denominator by its numerator, and so on.

$$E.g.,\quad \frac{79}{248}=\frac{1}{3\frac{11}{79}}=\frac{1}{3+\frac{1}{7\frac{2}{11}}}=\frac{1}{3+\frac{1}{7+\frac{1}{5\frac{1}{2}}}}=\frac{1}{3+\frac{1}{7+\frac{1}{5+\frac{1}{2}}}},$$

and its convergents are:

$$\frac{1}{3};\ \frac{1}{3+\frac{1}{7}},=\frac{7}{22};\ \frac{1}{3+\frac{1}{7+\frac{1}{5}}},=\frac{36}{113};\ \frac{1}{3+\frac{1}{7+\frac{1}{5+\frac{1}{2}}}},=\frac{79}{248},$$

whereof the last is the original fraction.

NOTE. The reader may find the h. c. msr. of 79 and 248. He will observe that the divisions made above in converting the common fraction into a continued fraction are precisely the same as those made in getting the h. c. msr. of the numerator and denominator, and that the several quotients are the partial denominators of the continued fraction. He will find this statement to contain a convenient working rule for converting common fractions into continued fractions.

So the value of π, the ratio of the circumference of a circle to its diameter, is 3.14159 26535 .... If in place of this endless decimal 3.1416 be used, then

$$\pi = 3\tfrac{1416}{10000} = 3 + \frac{1}{7\tfrac{88}{1416}} = 3 + \frac{1}{7+}\,\frac{1}{16\tfrac{8}{88}} = 3 + \frac{1}{7+}\,\frac{1}{16+}\,\frac{1}{11},$$

and its convergents are:

$$3;\quad 3 + \frac{1}{7}, = 3\tfrac{1}{7}, = \frac{22}{7};\quad 3 + \frac{1}{7+}\,\frac{1}{16}, = 3\tfrac{16}{113}, = \frac{355}{113};\ \cdots.$$

NOTE. The real value of π is incommensurable; but if the decimal be taken to 20 places, then the partial denominators are 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, ..., the continued fraction is

$$3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{292+\cdots},$$

and its convergents are:

$$3,\ 3\tfrac{1}{7},\ 3\tfrac{15}{106},\ 3\tfrac{16}{113},\ 3\tfrac{4687}{33102},\ \cdots.$$

(*b*) *Numerical or literal*, $n_1, n_2, \cdots$ *any entire numbers:*

Reduce to a proper fraction or mixed number; divide as above (a), *except that factors may be stricken out of the numerators* (divisors) *or introduced into the denominators* (dividends) *and reserved as partial numerators of the continued fraction.*

For the $\left\{\begin{array}{l}\text{striking out}\\ \text{introduction}\end{array}\right.$ of a factor $\left\{\begin{array}{l}\text{from a divisor}\\ \text{into a dividend}\end{array}\right.$ is equivalent to dividing both terms of the fractional part by the ratio of its numerator to this factor.

E.g., $$\frac{26}{101} = \frac{2}{7\tfrac{10}{13}} = \frac{2}{7}\,\frac{2}{2\tfrac{3}{5}} = \frac{2}{7}\,\frac{2}{2}\,\frac{2}{3}\,\frac{1}{3},$$

wherein the factor 2, stricken out from the first two divisors, 26 and 10, and introduced into the third dividend 5, becomes the numerator of the first three partial fractions;

or, both terms of $\tfrac{26}{101}$ are divided by 13, those of $\tfrac{10}{13}$ by 5, and those of $\tfrac{3}{5}$ by $\tfrac{3}{2}$.

So $$\frac{630}{2111}=\frac{2}{7-\frac{94}{315}}=\frac{2}{7-\frac{2}{7-\frac{14}{47}}}=\frac{2}{7-\frac{2}{7-\frac{2}{7-\frac{2}{7}}}},$$

wherein the divisors are 315, 47, 7, the reserved factors are all 2, and the quotients are so taken that the remainders are negative.

So $$\frac{24x+6x^2}{24+18x+x^2}=\frac{x}{1+\frac{12x+x^2}{24+6x}}=\frac{x}{1+\frac{x}{2+\frac{4x}{12+x}}}=\frac{x}{1+\frac{x}{2+\frac{x}{3+\frac{x}{4}}}},$$

wherein, at each division, the factor x has been stricken out of the divisor and reserved as the numerator of a partial fraction;

and the convergents are:

$$x;\quad \frac{x}{1+\frac{x}{2}},=\frac{2x}{2+x};\quad \frac{x}{1+\frac{x}{2+\frac{x}{3}}},=\frac{6x+x^2}{6+4x};\ \ldots.$$

In this example, if x be small, the successive convergents rapidly approach the true value of the fraction.

So $$\frac{\begin{array}{r|l} ax^3+ac & x \\ +ad & \end{array}}{\begin{array}{r|l} x^4+b & x^2+bd \\ +c & \\ +d & \end{array}}=\frac{a}{x+\frac{bx^2+bd}{\begin{array}{r|l} x^3+c & x \\ +d & \end{array}}}=\frac{a}{x+\frac{b}{x+\frac{cx}{x^2+d}}}=\frac{a}{x+\frac{b}{x+\frac{c}{x+\frac{d}{x}}}},$$

wherein, at the successive divisions, the factors a, b, c, d have been stricken out of the divisors and reserved as the numerators of the partial fractions;

and its convergents are:

$$\frac{a}{x},\quad \frac{ax}{x^2+b},\quad \frac{ax^2+ac}{\begin{array}{r|l} x^3+b & x \\ +c & \end{array}},\quad \ldots.$$

In this example, if x be large, the successive convergents rapidly approach the true value of the fraction.

NOTE. The continued fractions presented in this problem are all finite, and the original fraction is the last convergent; those which follow are infinite.

§ 3. CONVERSION OF SURDS.

Prob. 2. To convert a surd of the form $\sqrt{(a^2 \pm b)}$ into a continued fraction.

Let $x \equiv$ the value of $\sqrt{(a^2 \pm b)}$;

then $\because$ $x^2 = a^2 \pm b$, whence $x^2 - a^2 = \pm b$ and $x - a = \dfrac{\pm b}{a+x}$,

1] $$\therefore\ x = a \pm \frac{b}{a+x} = a \pm \cfrac{b}{a + a \pm \cfrac{b}{a+x}} = a \pm \cfrac{b}{2a \pm \cfrac{b}{a+x}}$$
$$= \cdots = a \pm \frac{b}{2a\pm}\,\frac{b}{2a\pm}\cdots,$$

and its convergents, if the entire number a be included, are:

$$a;\ a \pm \frac{b}{2a}, = \frac{2a^2 \pm b}{2a};\ a \pm \cfrac{b}{2a \pm \cfrac{b}{2a}}, = \frac{4a^3 \pm 3ab}{4a^2 \pm b};\ \cdots.$$

E.g., $\sqrt{2} = \sqrt{(1+1)} = 1 + \dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{2+}\cdots,$

and the convergents are:

$$1,\ 1\tfrac{1}{2},\ 1\tfrac{2}{5},\ 1\tfrac{5}{12},\ \cdots,\ = 1,\ \frac{3}{2},\ \frac{7}{5},\ \frac{17}{12},\ \cdots,$$

whose squares are:

$$1,\ \frac{9}{4},\ \frac{49}{25},\ \frac{289}{144},\ \cdots,\ = 2-1,\ 2+\frac{1}{4},\ 2-\frac{1}{25},\ 2+\frac{1}{144},\ \cdots.$$

So $\sqrt{3} = \sqrt{(4-1)} = 2 - \dfrac{1}{4-}\,\dfrac{1}{4-}\,\dfrac{1}{4-}\cdots,$

and the convergents are:

$$2,\ \frac{7}{4},\ \frac{26}{15},\ \frac{97}{56},\ \cdots,$$

whose squares are:

$$4,\ \frac{49}{16},\ \frac{676}{225},\ \frac{9409}{3136},\ \cdots,$$
$$= 3+1,\ 3+\frac{1}{16},\ 3+\frac{1}{225},\ 3+\frac{1}{3136},\ \cdots.$$

So $\sqrt{7} = \sqrt{(4+3)} = 2 + \dfrac{3}{4+}\,\dfrac{3}{4+}\,\dfrac{3}{4+}\cdots,$

and the convergents are:

$$2,\ 2\tfrac{3}{4},\ 2\tfrac{12}{19},\ 2\tfrac{57}{88},\ \cdots,\ = 2,\ \frac{11}{4},\ \frac{50}{19},\ \frac{233}{88},\ \cdots,$$

whose squares are:

$$4, \frac{121}{16}, \frac{2500}{361}, \frac{54289}{7744}, \cdots, =7-3,\ 7+\frac{9}{16},\ 7-\frac{27}{361},\ 7+\frac{81}{7744}, \cdots$$

Or $\sqrt{7}=\sqrt{(9-2)}=3-\cfrac{2}{6-\cfrac{2}{6-\cfrac{2}{6-\cdots}}}, \quad =3-\cfrac{1}{3-\cfrac{1}{6-\cfrac{1}{3-\cdots}}},$

and the convergents are:

$$3,\ \frac{16}{6},\ \frac{90}{34},\ \frac{508}{192}, \cdots, =3,\ \frac{8}{3},\ \frac{45}{17},\ \frac{127}{48}, \cdots,$$

whose squares are:

$$9, \frac{64}{9}, \frac{2025}{289}, \frac{16129}{2304}, \cdots, =7+2,\ 7+\frac{1}{9},\ 7+\frac{2}{289},\ 7+\frac{1}{2304}, \cdots.$$

NOTE 1. The rule is given in formula; the reader may translate it into words. In general, he will find any such formula translatable both as a theorem and as a rule. The first is a statement of facts and is put in the indicative mood; the other is a direction, an order, and is put in the imperative mood.

NOTE 2. If $\frac{b}{a^2}$ be small, the errors of the squares of the successive convergents, and therefore of the convergents themselves, diminish rapidly.

For $(a)^2, =a^2, \sim(a^2 \pm b)=b$.

So $\left(\frac{2a^2 \pm b}{2a}\right)^2, =\frac{4a^4 \pm 4a^2b+b^2}{4a^2}=a^2 \pm b+\frac{b^2}{4a^2},$

$$\sim(a^2 \pm b)=\frac{b^2}{4a^2}=b \cdot \frac{b}{4a^2}.$$

So $\left(\frac{4a^3 \pm 3ab}{4a^2 \pm b}\right)^2, =\frac{16a^6 \pm 24a^4b+9a^2b^2}{16a^4 \pm 8a^2b+b^2}$

$$=a^2 \pm b \mp \frac{b^3}{16a^4 \pm 8a^2b+b^2},$$

$$\sim(a^2 \pm b)=\frac{b^3}{16a^4 \pm 8a^2b+b^2}=b \cdot \left(\frac{b}{4a^2}\right)^2 : \left(1 \pm \frac{b}{4a^2}\right)^2.$$

So in the above numerical examples, the smaller $\frac{b}{a^2}$ is, the more rapidly does the series converge.

E.g., the series got by taking $\sqrt{7}=\sqrt{(9-2)}$, wherein $\frac{b}{a^2}=\frac{2}{9}$, converges much more rapidly than that got by taking $\sqrt{7}=\sqrt{(4+3)}$, wherein $\frac{b}{a^2}=\frac{3}{4}$.

So $\sqrt{3}=\sqrt{(4-1)}$ gives $\frac{b}{a^2}=\frac{1}{4}$, and the square of the fourth convergent differs from 3 by only $\frac{1}{3136}$;

whereas $\sqrt{2}=\sqrt{(1+1)}$ gives $\frac{b}{a^2}=\frac{1}{1}$, and the square of the fourth convergent differs from 2 by $\frac{1}{144}$.

NOTE 3. Another conversion may be made thus:

$$\because\ x^2-a^2=\pm b, \qquad \text{whence } x+a=\frac{\mp b}{a-x},$$

2] $$\therefore\ x=-a\mp\frac{b}{a-x}=-a\mp\cfrac{b}{a+a\pm\cfrac{b}{a-x}}$$

$$=-a\mp\cfrac{b}{2a\pm\cfrac{b}{a-x}}=\cdots=-a\mp\cfrac{b}{2a\pm\cfrac{b}{2a\pm\cdots}},$$

wherein the convergents are the opposites, each of each, of those found by the first conversion;

i.e., by the first process the convergents of the positive root were found, and by the second, those of the negative root, equally large but of contrary sense.

E.g., $\sqrt{3}=\sqrt{(4-1)}=-2+\cfrac{1}{4-\cfrac{1}{4-\cfrac{1}{4-\cdots}}}$,

and the convergents are

$$-2,\ -\frac{7}{4},\ -\frac{26}{15},\ -\frac{97}{56},\ \cdots.$$

☞ For other uses of continued fractions see the computation of logarithms [IX. § 3], and the solution of quadratic equations [XI. § 13].

§ 4. COMPUTATION OF CONVERGENTS.

PROB. 3. TO FIND THE CONVERGENTS OF A CONTINUED FRACTION.

FIRST METHOD. *For the first convergent reject all after the first partial fraction, for the second convergent reject all after the second partial fraction, and so on; reduce to simple fractions the complex fractions that remain.*

The examples given above have all been solved by this method.

SECOND METHOD. *Form two series,* $\begin{cases} N_1, N_2, N_3, \cdots N_k, \\ D_1, D_2, D_3, \cdots D_k, \end{cases}$

wherein $\begin{cases} N_1 \equiv n_1, & N_2 \equiv n_1 d_2 \qquad\quad = N_1 \cdot d_2, \\ D_1 \equiv d_1, & D_2 \equiv d_1 d_2 + n_2 = D_1 \cdot d_2 + n_2, \end{cases}$

$$\begin{cases} N_3 \equiv n_1 d_2 d_3 + n_1 n_3 \qquad\qquad = N_2 \cdot d_3 + N_1 \cdot n_3, \cdots \\ D_3 \equiv d_1 d_2 d_3 + n_2 d_3 + d_1 n_3 = D_2 \cdot d_3 + D_1 \cdot n_3, \cdots \end{cases}$$

3] $$\begin{cases} N_k \equiv N_{k-1} \cdot d_k + N_{k-2} \cdot n_k, \\ D_k \equiv D_{k-1} \cdot d_k + D_{k-2} \cdot n_k; \end{cases}$$

then are $\frac{N_1}{D_1}, \frac{N_2}{D_2}, \frac{N_3}{D_3}, \cdots \frac{N_k}{D_k}, \equiv \frac{N_{k-1} \cdot d_k + N_{k-2} \cdot n_k}{D_{k-1} \cdot d_k + D_{k-2} \cdot n_k},$

the 1*st*, 2*d*, 3*d*, ⋯ k*th*, *convergents.*

The reader may translate this formula into words.

1. *The law is true for the third convergent.*

For the first three convergents, by the first method, are

$$\frac{n_1}{d_1}, \quad \frac{n_1 d_2}{d_1 d_2 + n_2}, \quad \text{and} \quad \frac{n_1 d_2 \cdot d_3 + n_1 \cdot n_3}{d_1 d_2 \cdot d_3 + n_2 \cdot d_3 + d_1 \cdot n_3},$$

$$= \frac{N_2 \cdot d_3 + N_1 \cdot n_3}{D_2 \cdot d_3 + D_1 \cdot d_3}, \text{ as above.} \qquad \text{Q. E. D.}$$

2. *If the law be true for the* k*th convergent, it is true also for the* (k + 1)*th convergent.*

For $\because \frac{N_k}{D_k}, \equiv \cfrac{n_1}{d_1 + \cfrac{n_2}{d_2 + \cdots + \cfrac{n_k}{d_k}}}, \equiv \frac{N_{k-1} \cdot d_k + N_{k-2} \cdot n_k}{D_{k-1} \cdot d_k + D_{k-2} \cdot n_k},$ [hyp.

an identity, whatever expression or value d_k may stand for, and therefore an identity when d_k is replaced by $d_k + \frac{n_{k+1}}{d_{k+1}}$,

$$\therefore \frac{N_{k+1}}{D_{k+1}}, \equiv \frac{n_1}{d_1+}\frac{n_2}{d_2+}\cdots+\frac{n_k}{d_k+}\frac{n_{k+1}}{d_{k+1}},$$

$$= \frac{N_{k-1}\cdot\left(d_k+\dfrac{n_{k+1}}{d_{k+1}}\right)+N_{k-2}\cdot n_k}{D_{k-1}\cdot\left(d_k+\dfrac{n_{k+1}}{d_{k+1}}\right)+D_{k-2}\cdot n_k}$$

$$= \frac{N_{k-1}\cdot d_k\cdot d_{k+1}+N_{k-1}\cdot n_{k+1}+N_{k-2}\cdot n_k\cdot d_{k+1}}{D_{k-1}\cdot d_k\cdot d_{k+1}+D_{k-1}\cdot n_{k+1}+D_{k-2}\cdot n_k\cdot d_{k+1}}$$

$$= \frac{(N_{k-1}\cdot d_k+N_{k-2}\cdot n_k)\cdot d_{k+1}+N_{k-1}\cdot n_{k+1}}{(D_{k-1}\cdot d_k+D_{k-2}\cdot n_k)\cdot d_{k+1}+D_{k-1}\cdot n_{k+1}}$$

$$= \frac{N_k\cdot d_{k+1}+N_{k-1}\cdot n_{k+1}}{D_k\cdot d_{k+1}+D_{k-1}\cdot n_{k+1}}. \qquad \text{Q. E. D.}$$

3. *The law is true whatever the index* **k**.

For $\because$ it is true for $k=3$, [1
$\therefore$ it is true for $k=4$. [2

So for $k=5$; for $k=6$; for $k=7$; and so on. Q. E. D.

E.g., of the fraction $\dfrac{a}{x+}\dfrac{b}{x+}\dfrac{c}{x+}\dfrac{d}{x+\cdots}$,
the convergents are:

$$\frac{a}{x};\quad \frac{ax}{x^2+b};\quad \frac{ax\cdot x+a\cdot c}{(x^2+b)\cdot x+x\cdot c}, = \frac{ax^2+ac}{x^3+(b+c)x};$$

$$\frac{(ax^2+ac)\cdot x+ax\cdot d}{(x^3+(b+c)x)\cdot x+(x^2+b)\cdot d}, = \frac{ax^3+(ac+ad)x}{x^4+(b+c+d)x^2+bd};\ \cdots.$$

So of the fraction $\dfrac{x}{a+}\dfrac{x}{b+}\dfrac{x}{c+\cdots}$,
the convergents are:

$$\frac{x}{a},\ \frac{bx}{x+ab},\ \frac{x^2+bcx}{(a+c)x+abc},\ \frac{(b+d)x^2+bcdx}{x^2+(ab+ad+cd)x+abcd};\ \cdots.$$

In particular, when $n_1, n_2, \cdots n_k$, *each* $=1$, *then the formula becomes:*

4] $$\frac{N_k}{D_k}=\frac{N_{k-1}\cdot d_k+N_{k-2}}{D_{k-1}\cdot d_k+D_{k-2}}.$$

The reader may translate this formula into words. He may also demonstrate it anew, putting 1 for $n_1, n_2, \cdots n_k, n_{k+1}$, in the general demonstration.

E.g., of the fraction $\dfrac{1}{a+}\dfrac{1}{b+}\dfrac{1}{c+\cdots}$,

the convergents are:

$$\frac{1}{a};\ \frac{b}{ab+1};\ \frac{b\cdot c+1}{(ab+1)\cdot c+a},=\frac{bc+1}{abc+a+c};\ \cdots.$$

Compare the result of the previous example, when $x=1$.

NOTE. Formulae [3, 4] may be made to include convergents of the mixed number $n_0+\dfrac{n_1}{d_1+}\dfrac{n_2}{d_2+\cdots}$, as follows:

Let $\begin{cases} N_{-1}\equiv 1, & N_0\equiv n_0, \\ D_{-1}\equiv 0, & D_0\equiv 1, \end{cases}$

wherein $n_0\equiv$ any number, perhaps 0;

then will $\begin{cases} N_1=N_0\cdot d_1+N_{-1}\cdot n_1, & N_2=N_1\cdot d_2+N_0\cdot n_2, \\ D_1=D_0\cdot d_1+D_{-1}\cdot n_1, & D_2=D_1\cdot d_2+D_0\cdot n_2, \end{cases}$

and $\dfrac{N_1}{D_1}=\dfrac{N_0\cdot d_1+N_{-1}\cdot n_1}{D_0\cdot d_1+D_{-1}\cdot n_1}$, $\quad\dfrac{N_2}{D_2}=\dfrac{N_1\cdot d_2+N_0\cdot n_2}{D_1\cdot d_2+D_0\cdot n_2}$,

as the reader may verify;

but it is convenient to speak of $\dfrac{N_1}{D_1},=n_0+\dfrac{n_1}{d_1}=\dfrac{n_0 d_1+n_1}{d_1}$, as the 1st convergent, even when n_0 has a value not 0;

and of $\dfrac{N_0}{D_0}$ and $\dfrac{N_{-1}}{D_{-1}}$ as the 0th and ⁻1st convergents.

E.g., of π, $=3\dfrac{1}{7}\dfrac{1}{15}\dfrac{1}{1\cdots}$, the convergents after $\dfrac{N_0}{D_0},=\dfrac{3}{1}$, are

$$\frac{3\cdot 7+1}{1\cdot 7+0},=\frac{22}{7};\ \frac{22\cdot 15+3}{7\cdot 15+1},=\frac{333}{106};\ \frac{333\cdot 1+22}{106\cdot 1+7},=\frac{355}{113};$$

$\cdots$; as in § 2.

§ 5. GENERAL PROPERTIES.

In this section, let $v_1, v_2, v_3, \cdots$ be the true values to which

$$c_k, \equiv c_{1,k}, \equiv \frac{N_k}{D_k}, \equiv \frac{n_1}{d_1 +} \frac{n_2}{d_2 +} \frac{n_3}{d_3 + \cdots} \frac{n_k}{d_k},$$

$$c_{2,k}, \equiv \frac{N_{2,k}}{D_{2,k}}, \equiv \frac{n_2}{d_2 +} \frac{n_3}{d_3 + \cdots} \frac{n_k}{d_k},$$

$$c_{3,k}, \equiv \frac{N_{3,k}}{D_{3,k}}, \equiv \frac{n_3}{d_3 + \cdots} \frac{n_k}{d_k},$$

$$\cdots \quad \cdots \quad \cdots$$

are convergents; whence

$$c_k = \frac{n_1}{d_1 + c_{2,k}} = \frac{n_1}{d_1 +} \frac{n_2}{d_2 + c_{3,k}} = \cdots,$$

$$v_1 = \frac{n_1}{d_1 + v_2} = \frac{n_1}{d_1 +} \frac{n_2}{d_2 + v_3} = \cdots.$$

Theor. 1. *If the partial numerators and denominators be all positive, the convergents $c_{1,1}, c_{1,2}, \cdots$ are alternately greater and less than the true value v_1.*

Let $n_1, n_2, \cdots d_1, d_2, \cdots$ be all positive; then will

$$c_1 > v_1, \quad c_2 < v_1, \quad c_3 > v_1, \cdots c_k \left\{ \begin{matrix} > \\ < \end{matrix} \right\} v_1, \left\{ \begin{matrix} k \text{ odd.} \\ k \text{ even.} \end{matrix} \right.$$

For $\dfrac{n_1}{d_1} > \dfrac{n_1}{d_1 + v_2}$, [II. ax. 18

i.e., $c_1 > v_1$. Q. E. D.

So $\because \dfrac{n_2}{d_2} > \dfrac{n_2}{d_2 + v_3}$, $\therefore \dfrac{n_1}{d_1 +} \dfrac{n_2}{d_2} < \dfrac{n_1}{d_1 +} \dfrac{n_2}{d_2 + v_3}$, [II. ax. 18

i.e., $c_2 < v_1$. Q. E. D.

So $\because \dfrac{n_3}{d_3} > \dfrac{n_3}{d_3 + v_4}$, $\therefore \dfrac{n_2}{d_2 +} \dfrac{n_3}{d_3} < \dfrac{n_2}{d_2 +} \dfrac{n_3}{d_3 + v_4}$,

$$\therefore \frac{n_1}{d_1 +} \frac{n_2}{d_2 +} \frac{n_3}{d_3} > \frac{n_1}{d_1 +} \frac{n_2}{d_2 +} \frac{n_3}{d_3 + v_4},$$

i.e., $c_3 > v_1$. Q. E. D.

$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots,$

5] $\therefore c_k \left\{ \begin{matrix} > \\ < \end{matrix} \right\} v_1, \left\{ \begin{matrix} k \text{ odd.} \\ k \text{ even.} \end{matrix} \right.$ Q. E. D.

THEOR. 2. *If any convergent, c_k, be subtracted from the next following convergent, c_{k+1}, then will the remainder,*

6] $$\frac{N_{k+1}}{D_{k+1}}-\frac{N_k}{D_k}, =(-1)^k\cdot\frac{n_1\cdot n_2\cdots n_{k+1}}{D_k\cdot D_{k+1}}.$$

Since $\dfrac{N_{k+1}}{D_{k+1}}-\dfrac{N_k}{D_k}=\dfrac{D_k\cdot N_{k+1}-N_k\cdot D_{k+1}}{D_k\cdot D_{k+1}}$, it is to prove that

$$D_k\cdot N_{k+1}-N_k\cdot D_{k+1}=(-1)^k\cdot n_1\cdot n_2\cdots n_{k+1}.$$

1. *The law is true for* $k=1$.

For $\because\ N_1=n_1,\quad D_1=d_1,\quad N_2=n_1 d_2,\quad D_2=d_1 d_2+n_2,$

$\therefore\ D_1\cdot N_2-N_1\cdot D_2=d_1\cdot n_1 d_2-n_1\cdot(d_1 d_2+n_2)$

$=-n_1\cdot n_2=(-1)^1\cdot n_1\cdot n_2.$ Q. E. D.

2. *If* $D_{k-1}\cdot N_k-N_{k-1}\cdot D_k=(-1)^{k-1}\cdot n_1\cdot n_2\cdots n_k$, *then will* $D_k\cdot N_{k+1}-N_k\cdot D_{k+1}=(-1)^k\cdot n_1\cdot n_2\cdots n_{k+1}$.

For $D_k\cdot N_{k+1}-N_k\cdot D_{k+1}=D_k\cdot(N_k\cdot d_{k+1}+N_{k-1}\cdot n_{k+1})$

$-N_k\cdot(D_k\cdot d_{k+1}+D_{k-1}\cdot n_{k+1})$ [3

$=(N_{k-1}\cdot D_k-D_{k-1}\cdot N_k)\cdot n_{k+1}$

$=-(D_{k-1}\cdot N_k-N_{k-1}\cdot D_k)\cdot n_{k+1}$

$=-(-1)^{k-1}\cdot n_1\cdot n_2\cdots n_k\cdot n_{k+1}$ [hyp.

$=(-1)^k\cdot n_1\cdot n_2\cdots n_{k+1}.$ Q. E. D.

3. *The law is true whatever the index* k.

For $\because$ it is true for $k=1$, [1, above

$\therefore$ it is true for $k=2$. [2, above

So for $k=3$, for $k=4$, for $k=5$, $\cdots$. Q. E. D.

In particular: If $n_1, n_2, \cdots$ each $=1$, then will

7] $$c_{k+1}-c_k=\frac{(-1)^k}{D_k\cdot D_{k+1}}.$$

COR. 1. *The error of any convergent,* $\dfrac{N_k}{D_k}$, *of* $\dfrac{1}{d_1+}\dfrac{1}{d_2+}\cdots$, *when* $d_1, d_2, \cdots$ *are all positive, is less than* $\dfrac{1}{D_k\cdot D_{k+1}}$, *and much more is it less than* $\dfrac{1}{D_k^2}$.

For $\because$ the true value lies between $\dfrac{N_k}{D_k}$ and $\dfrac{N_{k+1}}{D_{k+1}}$, [th. 1

and $\because$ these differ by only $\dfrac{1}{D_k\cdot D_{k+1}}$, $<\dfrac{1}{D_k^2}$,

$\therefore$ etc. Q. E. D.

COR. 2. $c_k = \frac{n_1}{D_1} - \frac{n_1 \cdot n_2}{D_1 \cdot D_2} + \frac{n_1 \cdot n_2 \cdot n_3}{D_2 \cdot D_3} - \cdots + (-1)^{k-1} \cdot \frac{n_1 \cdot n_2 \cdots n_k}{D_{k-1} \cdot D_k}$.

For $\because c_1 = \frac{n_1}{D_1}$,

$$c_2 - c_1 = -\frac{n_1 \cdot n_2}{D_1 \cdot D_2}, \qquad [\text{th.}$$

$$c_3 - c_2 = \frac{n_1 \cdot n_2 \cdot n_3}{D_2 \cdot D_3},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots,$$

and $c_k - c_{k-1} = (-1)^{k-1} \frac{n_1 \cdot n_2 \cdots n_k}{D_{k-1} D_k}$;

8] $\therefore c_k = \frac{n_1}{D_1} - \frac{n_1 \cdot n_2}{D_1 \cdot D_2} + \frac{n_1 \cdot n_2 \cdot n_3}{D_2 \cdot D_3} - \cdots + (-1)^{k-1} \cdot \frac{n_1 \cdot n_2 \cdots n_k}{D_{k-1} \cdot D_k}$.

In particular: If $n_1, n_2, \cdots$ each $= 1$, then will

9] $c_k = \frac{1}{D_1} - \frac{1}{D_1 \cdot D_2} + \frac{1}{D_2 \cdot D_3} - \cdots + \frac{(-1)^{k-1}}{D_{k-1} \cdot D_k}$.

NOTE. This formula gives a rule by which any continued fraction may be reduced to a series.

E.g., $\sqrt{28} = \sqrt{(25+3)} = 5 + \frac{3}{10+} \frac{3}{10+\cdots}$,

whose 1st, 2d, $\cdots$ convergents have the denominators

10, 103, 1060, 10909, 112270, $\cdots$;

$$\therefore \sqrt{28} = 5 + \frac{3}{10} - \frac{3^2}{10 \cdot 103} + \frac{3^3}{103 \cdot 1060} - \frac{3^4}{1060 \cdot 10909} + \cdots,$$

a series whose successive terms grow smaller very rapidly.

COR. 3. *If* $n_1, n_2, \cdots d_1, d_2, \cdots$ *be all positive, the successive differences grow smaller and smaller.*

For $\because c_{k-1} \sim c_k = \frac{n_1 n_2 \cdots n_k}{D_{k-1} \cdot D_k}$ and $c_k \sim c_{k+1} = \frac{n_1 n_2 \cdots n_k n_{k+1}}{D_k D_{k+1}}$, [cr.1

and $\because \frac{n_1 \cdot n_2 \cdots n_k}{D_{k-1} D_k} : \frac{n_1 \cdot n_2 \cdots n_k \cdot n_{k+1}}{D_k \cdot D_{k+1}} = \frac{D_{k+1}}{D_{k-1} \cdot n_{k+1}}$

$$= \frac{D_k \cdot d_{k+1} + D_{k-1} \cdot n_{k+1}}{D_{k-1} \cdot n_{k+1}} \qquad [3$$

$$= 1 + \frac{D_k \cdot d_{k+1}}{D_{k-1} \cdot n_{k+1}}, \quad > 1;$$

$\therefore c_{k-1} \sim c_k > c^k \sim c_{k+1}$. Q. E. D.

COR. 4. *The successive convergents approach to each other, and therefore to the true value which lies between them, rapidly when the ratios* $d_1 \cdot d_2 : n_2,\ d_2 \cdot d_3 : n_3, \cdots$ *are large.*

For $\dfrac{C_{k-1} \sim C_k}{C_k \sim C_{k+1}},\ = 1 + \dfrac{D_k \cdot d_{k+1}}{D_{k-1} \cdot n_{k+1}},$ [above

$$= 1 + \frac{(D_{k-1} \cdot d_k + D_{k-2} \cdot n_k)\, d_{k+1}}{D_{k-1} \cdot n_{k+1}}$$

$$= 1 + \frac{d_k \cdot d_{k+1}}{n_{k+1}} + \cdots.$$

E.g., the convergents to $\sqrt{(a^2+b)},\ = a + \dfrac{b}{2a+}\,\dfrac{b}{2a+\cdots},$ approach the true value rapidly when $\dfrac{b}{a^2}$ is small. [§ 2

COR. 5. *If* $n_1, n_2, \cdots$ *each* $= 1$, *and* $d_1, d_2, \cdots$ *be all entire, all the convergents are simple fractions in their lowest terms, and their consecutive* $\left\{\begin{array}{l}\textit{numerators}\\ \textit{denominators}\end{array}\right.$ *are prime to each other.*

For $\because$ every common measure of N_k, D_k is a measure of

$N_k \cdot D_{k+1} \sim D_k \cdot N_{k+1},\ = 1.$ [III. th. 2 cr. 4

$\therefore$ the h. c. msr. of N_k, D_k is 1. Q. E. D.

So of N_k, N_{k+1}, and of D_k, D_{k+1}.

COR. 6. *If* $n_1, n_2, \cdots$ *each* $= 1$, *and* $d_1, d_2, \cdots$ *be positive, then between two consecutive convergents there lies no fraction whose denominator is smaller than the largest of their denominators.*

For, let $\dfrac{N}{D},\ \neq \dfrac{N_k}{D_k},\ \equiv$ any simple fraction wherein $D < D_{k+1}$;

then $\because\ \dfrac{N}{D} \sim \dfrac{N_k}{D_k} = \dfrac{N \cdot D_k \sim D \cdot N_k}{D \cdot D_k},$

whose numerator, being entire and not 0, [II. ax. 23, hyp.

either $\equiv$ or > 1,

and whose denominator $< D_k \cdot D_{k+1}$, [hyp.

$\therefore\ \dfrac{N}{D} \sim \dfrac{N_k}{D_k} > \dfrac{1}{D_k \cdot D_{k+1}};$ [II. ax. 17, 18

i.e., $\dfrac{N}{D}$ differs more from $\dfrac{N_k}{D_k}$ than $\dfrac{N_{k+1}}{D_{k+1}}$ does,

and cannot lie between them. Q. E. D.

Theor. 3. *The difference* $\frac{\mathrm{N}_{k+2}}{\mathrm{D}_{k+2}}-\frac{\mathrm{N}_k}{\mathrm{D}_k}=\frac{(-1)^k\,\mathrm{n}_1\cdots\mathrm{n}_{k+1}\cdot\mathrm{d}_{k+2}}{\mathrm{D}_k\cdot\mathrm{D}_{k+2}}.$

For $\because\ \frac{\mathrm{N}_{k+2}}{\mathrm{D}_{k+2}}-\frac{\mathrm{N}_k}{\mathrm{D}_k}=\frac{\mathrm{D}_k\cdot\mathrm{N}_{k+2}-\mathrm{N}_k\cdot\mathrm{D}_{k+2}}{\mathrm{D}_k\cdot\mathrm{D}_{k+2}},$

$$\begin{aligned}\text{whose numerator} &= \mathrm{D}_k\cdot(\mathrm{N}_{k+1}\cdot d_{k+2}+\mathrm{N}_k\cdot n_{k+2})\\ &\quad-\mathrm{N}_k\cdot(\mathrm{D}_{k+1}\cdot d_{k+2}+\mathrm{D}_k\cdot n_{k+2})\\ &= (\mathrm{D}_k\cdot\mathrm{N}_{k+1}-\mathrm{N}_k\cdot\mathrm{D}_{k+1})\cdot d_{k+2}\\ &=(-1)^k\cdot n_1\cdots n_{k+1}\cdot d_{k+2}, \qquad [\text{th. } 2\end{aligned}$$

$\therefore$ etc. Q. E. D.

Cor. 1. *If* $\mathrm{n}_1, \mathrm{n}_2, \cdots$ *and* $\mathrm{d}_1, \mathrm{d}_2, \cdots$ *be all positive,*

then $\mathrm{c}_1 > \mathrm{c}_3 > \mathrm{c}_5 > \cdots,$

and $\mathrm{c}_2 < \mathrm{c}_4 < \mathrm{c}_6 < \cdots.$

Cor. 2. $$\mathrm{v}_1=\mathrm{n}_0+\frac{\mathrm{n}_1\mathrm{d}_2}{1\cdot\mathrm{D}_2}+\frac{\mathrm{n}_1\mathrm{n}_2\mathrm{n}_3\mathrm{d}_4}{\mathrm{D}_2\cdot\mathrm{D}_4}+\frac{\mathrm{n}_1\cdots\mathrm{n}_5\mathrm{d}_6}{\mathrm{D}_4\cdot\mathrm{D}_6}+\cdots$$
$$=\frac{\mathrm{N}_1}{\mathrm{D}_1}-\frac{n_1n_2\mathrm{d}_3}{\mathrm{D}_1\cdot\mathrm{D}_3}-\frac{n_1\cdots n_4 d_5}{\mathrm{D}_3\cdot\mathrm{D}_5}-\cdots.$$

Theor. 4. *In any continued fraction*

$$\mathrm{v}_1-\mathrm{c}_k=\frac{(-1)^k\cdot\mathrm{n}_1\cdots\mathrm{n}_{k+1}}{\mathrm{D}_k\cdot(\mathrm{D}_{k+1}+\mathrm{D}_k\cdot\mathrm{v}_{k+2})}.$$

For $\because\ \mathrm{c}_{k+1}=\frac{n_1}{d_1+}\cdots\frac{n_{k+1}}{d_{k+1}},$ and $\mathrm{v}_1=\frac{n_1}{d_1+}\cdots\frac{n_{k+1}}{d_{k+1}+\mathrm{v}_{k+2}},$

$\therefore$ v_1 is what c_{k+1} becomes when for d_{k+1} is put $\overline{d_{k+1}+\mathrm{v}_{k+2}},$

and $\because\ \mathrm{c}_{k+1}-\mathrm{c}_k=\frac{(-1)^k\cdot n_1\cdots n_{k+1}}{\mathrm{D}_k\cdot\mathrm{D}_{k+1}}$

$$=\frac{(-1)^k\cdot n_1\cdots n_{k+1}}{\mathrm{D}_k\cdot(\mathrm{D}_k\cdot d_{k+1}+\mathrm{D}_{k-1}\cdot n_{k+1})},$$

$$\therefore\ \mathrm{v}_1-\mathrm{c}_k=\frac{(-1)^k\cdot n_1\cdots n_{k+1}}{\mathrm{D}_k\cdot(\mathrm{D}_k\cdot\overline{d_{k+1}+\mathrm{v}_{k+2}}+\mathrm{D}_{k-1}\cdot n_{k+1})}$$

$$=\frac{(-1)^k\cdot n_1\cdots n_{k+1}}{\mathrm{D}_k\cdot(\mathrm{D}_{k+1}+\mathrm{D}_k\cdot\mathrm{v}_{k+2})}.\qquad \text{Q. E. D.}$$

COR. 1. *If* $n_1, n_2, \cdots$ *each* $=1$, *and* $d_{k+2}, d_{k+3}, \cdots$ *each* $\nless 1$,

then $\quad v_1 \sim c_k$ *lies between* $\dfrac{1}{D_k \cdot D_{k+1}}$ *and* $\dfrac{1}{D_k \cdot (D_{k+1} + D_k)}$.

For $\because v_{k+2}, = \dfrac{n_{k+2}}{d_{k+2} + \cdots}$, then lies between 0 and 1,

$\therefore D_k \cdot (D_{k+1} + D_k \cdot v_{k+2})$, the denominator of $v_1 \sim c_k$, lies between $D_k \cdot D_{k+1}$ and $D_k \cdot (D_{k+1} + D_k)$;

$\therefore$ etc. Q. E. D.

NOTE. So, if of $n_1, n_2, \cdots$ any are negative but all $\equiv 1$, and if $d_1, d_2, \cdots \nless 2$, then v_{k+2} lies between -1 and $+1$ inclusive, and $v_1 \sim c_k$ between $\dfrac{1}{D_k \cdot (D_{k+1} - D_k)}$ and $\dfrac{1}{D_k \cdot (D_{k+1} + D_k)}$.

COR. 2. *If* $n_1, n_2, \cdots$ *each* $=1$, *and* $d_1, d_2, \cdots$ *each* $\nless 1$, *every convergent differs less than the previous one from the true value.*

For $\because D_{k+2} = D_{k+1} \cdot d_{k+2} + D_k \nless D_{k+1} + D_k$, $\quad$ [$d_{k+2} \nless 1$

and $\because D_{k+1} > D_k$,

$\therefore D_{k+1} \cdot D_{k+2} > D_k \cdot (D_{k+1} + D_k)$;

and $\because c_{k+1} \sim v_1 \ngtr \dfrac{1}{D_{k+1} \cdot D_{k+2}}$, $c_k \sim v_1 \nless \dfrac{1}{D_k \cdot (D_{k+1} + D_k)}$, $\quad$ [cr. 1

$\therefore c_{k+1} \sim v_1 < c_k \sim v_1$. Q. E. D.

PROB. 4. GIVEN AN INCOMMENSURABLE, x, AND AN INTEGER, i: TO FIND A SIMPLE FRACTION $\frac{N}{D}$ WHOSE DENOMINATOR IS SUCH THAT $D \ngtr i$ AND $\frac{N}{D} \sim x < \dfrac{1}{D \cdot i}$.

Reduce x *to a continued fraction wherein* $n_1, n_2, \cdots$ *are each* 1, *and* $d_1, d_2, \cdots$ *are all positive integers; find* $\frac{N_k}{D_k}$, *the highest convergent whose denominator* $\ngtr i$; *it is the fraction required.*

For $\because x \sim \dfrac{N_k}{D_k} < \dfrac{1}{D_k \cdot D_{k+1}}$, $\quad$ [th. 2 cr. 1

and $\because D_{k+1} > i$,

$\therefore x \sim \dfrac{N_k}{D_k} < \dfrac{1}{D_k \cdot i}$, $\quad$ *i.e.*, $< \dfrac{1}{D \cdot i}$. Q. E. D.

§ 6. SECONDARY CONVERGENTS.

If $n_1, n_2, \cdots$ each $=1$, and $d_1, d_2, \cdots$ be all positive integers, then the series $c_0, c_1, c_2, c_3, \cdots$ may be resolved into two series:

$c_0, c_2, c_4, \cdots$, all too small, and $c_1, c_3, c_5, \cdots$, all too large,

wherein $c_2 - c_0 = \dfrac{d_2}{D_0 \cdot D_2}, \quad c_4 - c_2 = \dfrac{d_4}{D_2 \cdot D_4}, \cdots,$ [th. 3

and $c_3 - c_1 = \dfrac{-d_3}{D_1 \cdot D_3}, \quad c_5 - c_3 = \dfrac{-d_5}{D_3 \cdot D_5}, \cdots.$

Put $1, 2, 3, \cdots d_2-1$, in turn, for d_2 in $\dfrac{N_1 \cdot d_2 + N_0}{D_1 \cdot d_2 + D_0}, = c_2,$

and $1, 2, 3, \cdots d_4-1$, in turn, for d_4 in c_4, and so on;

then a series of *secondary convergents* is formed, lying between the *primary convergents* $c_0, c_2, c_4, \cdots$;

and the whole series, whose terms are all too small, is:

$$c_0, \frac{N_1+N_0}{D_1+D_0}, \frac{2\cdot N_1+N_0}{2\cdot D_1+D_0}, \frac{3\cdot N_1+N_0}{3\cdot D_1+D_0}, \cdots \frac{(d_2-1)\cdot N_1+N_0}{(d_2-1)\cdot D_1+D_0}, c_2,$$

$$\frac{N_3+N_2}{D_3+D_2}, \frac{2\cdot N_3+N_2}{2\cdot D_3+D_2}, \frac{3\cdot N_3+N_2}{3\cdot D_3+D_2}, \cdots \frac{(d_4-1)\cdot N_3+N_2}{(d_4-1)\cdot D_3+D_2}, c_4;$$

or $c_0, c_{0|1}, c_{0|2}, \cdots c_{0|d_2-1}, c_2, c_{2|1}, c_{2|2}, \cdots c_{2|d_4-1}, c_4, \cdots.$

So, put $1, 2, 3, \cdots d_3-1$, in turn, for d_3 in c_3,

and $1, 2, 3, \cdots d_5-1$, in turn, for d_5 in c_5, and so on;

then a second series is formed, whose terms are all too large:

$$c_1, \frac{N_2+N_1}{D_2+D_1}, \frac{2\cdot N_2+N_1}{2\cdot D_2+D_1}, \cdots c_3, \frac{N_4+N_3}{D_4+D_3}, \frac{2\cdot N_4+N_3}{2\cdot D_4+D_3}, \cdots c_5,$$

or $c_1, c_{1|1}, c_{1|2}, \cdots c_{1|d_3-1}, c_3, c_{3|1}, c_{3|2}, \cdots c_{3|d_5-1}, c_5, \cdots.$

Theor. 5. *The terms of the first series* $c_0, c_{0|1}, c_{0|2}, \cdots$ *grow greater and greater; of the second series* $c_1, c_{1|1}, c_{1|2}, \cdots$ *less and less; and the differences of successive terms, smaller and smaller.*

For $c_{0|1} - c_0 = \dfrac{N_1+N_0}{D_1+D_0} - \dfrac{N_0}{D_0} = \dfrac{1}{D_0\cdot(D_1+D_0)},$ [th. 2

$c_{0|2} - c_{0|1} = \dfrac{1}{(D_1+D_0)\cdot(2\,D_1+D_0)},$ and so on.

So $c_{1|1} - c_1 = \dfrac{N_2+N_1}{D_2+D_1} - \dfrac{N_1}{D_1} = \dfrac{-1}{D_1\cdot(D_2+D_1)},$

$c_{1|2} - c_{1|1} = \dfrac{-1}{(D_2+D_1)\cdot(2\,D_2+D_1)},$ and so on.

THEOR. 6. $c_{k|r} \sim c_{k+1} = \dfrac{1}{D_{k|r} \cdot D_{k+1}}$, *wherein* $D_{k|r} \equiv r \cdot D_{k+1} + D_k$.

For, reduce $c_{k|r} \sim c_{k+1}$ to the common denominator $D_{k|r} \cdot D_k$;
then the numerator, $(r \cdot N_{k+1} + N_k) \cdot D_{k+1} \sim (r \cdot D_{k+1} + D_k) \cdot N_{k+1}$,
$= N_k \cdot D_{k+1} \sim D_k \cdot N_{k+1}$, $= 1$. Q. E. D.

COR. 1. $c_{k|r}$ *is in its lowest terms.*

For $\because$ every common measure of $N_{k|r}$, $D_{k|r}$ is a measure of $N_{k|r} \cdot D_{k+1} \sim D_{k|r} \cdot N_{k+1}$, $= 1$, [III. th. 2 cr. 4, th.
$\therefore$ the h. c. msr. of $N_{k|r}$, $D_{k|r}$ is 1. Q. E. D.

COR. 2. *Between* $c_{k|r}$ *and* v_1 *there lies no simple fraction with terms so small as those of* $c_{k|r}$.

For $\because$ $c_{k|r} \sim c_{k+1} = \dfrac{1}{D_{k|r} \cdot D_{k+1}}$,
$\therefore$ between $c_{k|r}$ and c_{k+1} there lies no simple fraction with denominator so small as $D_{k|r}$; [compare th. 2 cr. 5
and $\because$ v_1 lies between $c_{k|r}$ and c_{k+1},
$\therefore$ between $c_{k|r}$ and v_1 there lies no simple fraction with terms so small as those of $c_{k|r}$. Q. E. D.

THEOR. 7. $c_{k|r} \sim c_{k|s} = \dfrac{r \sim s}{D_{k|r} \cdot D_{k|s}}$.

For, reduce $c_{k|r} \sim c_{k|s}$ to the common denominator $D_{k|r} \cdot D_{k|s}$;
then the numerator,

$$(N_{k+1} \cdot r + N_k) \cdot (D_{k+1} \cdot s + D_k) \sim (D_{k+1} \cdot r + D_k) \cdot (N_{k+1} \cdot s + N_k),$$
$$= (N_k \cdot D_{k+1} \sim D_k \cdot N_{k+1}) \cdot (r \sim s) = r \sim s. \quad \text{Q. E. D. [th. 2}$$

THEOR. 8. *When* $r > \frac{1}{2} d_{k+2}$, *then* $c_{k|r}$ *differs less from* v_1 *than does any simple fraction with terms as small as those of* $c_{k|r}$.

For $\because$ $v_1 = \dfrac{N_k + (d_{k+1} + \cdots) \cdot N_{k+1}}{D_k + (d_{k+1} + \cdots) \cdot D_{k+1}}$, $\equiv c_{k|s}$, [$s < d_{k+2} + 1$

and $\because$ $c_{k|r} \sim c_{k|s} = \dfrac{r \sim s}{D_{k|r} \cdot D_{k|s}} < \dfrac{r}{(r \cdot D_{k+1} + D_k) \cdot D_{k|s}}$ [$r > \frac{1}{2} s$

$$< \frac{1}{D_{k+1} \cdot D_{k|s}} = c_{k+1} \sim c_{k|s},$$

$\therefore$ $c_{k|r}$ is nearer than c_{k+1} to v_1. Q. E. D.

§ 7. EXAMPLES.

§§ 2, 4. PROBS. 1, 3.

1. Convert the following fractions into continued fractions, and get five convergents for each of them, if there be so many:

$$\frac{47}{223},\quad \frac{293}{631},\quad \frac{839}{739},\quad \frac{995}{293},\quad \frac{1051}{2371},\quad \frac{2371}{4049}.$$

2. Find the limits of error of the fourth convergent of .1357, 2.7182818, .43429448, 180 : 57.2957795.

3. The true length of the equinoctial year is $365^d\,5^h\,48^m\,46^s$: reduce the ratio, $5^h\,48^m\,46^s : 24^h$, to a continued fraction, find five convergents, and thence show how often leap-year should come. Find the limits of error for the fifth convergent.

4. The earth makes one sidereal revolution about the sun in 365.2564 days, and Venus in 224.7008 days; how often will the two planets be in *conjunction* (in line with the sun)? Get four convergents, and the limits of error for the fourth.

§§ 3, 4. PROBS. 2, 3.

5. Convert the following surds into continued fractions, and get five convergents. Find the limits of error.

$$\surd 6,\ \surd 10,\ \surd 59,\ \surd 119;\quad \sqrt{\frac{5}{3}},\ \sqrt{\frac{3}{2}},\ \sqrt{\frac{3}{16}},\ \sqrt{\frac{16}{3}};$$
$$\surd(a^2-1),\ \surd(a^2+a),\ \surd(a^2-a),\ \surd(a^2+a+1).$$

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6. Express $\surd c, = \surd(m^2 + \overline{c-m^2})$, as a continued fraction, and show that whatever the numbers m, m',

$$m + \cfrac{c-m^2}{2m + \cfrac{c-m^2}{2m+\cdots}} = \pm m' \pm \cfrac{c-m'^2}{2m' + \cfrac{c-m'^2}{2m'+\cdots}}.$$

7. Develop into series the sixth convergents of [th. 2 cr. 2

$$3.14159265,\ \frac{3015}{6961},\ \surd 145,\ \sqrt{\frac{49}{50}};\ \surd(a^2+b),\ \surd(a^2-b).$$

8. Show that:

$$\frac{1}{m}\cdot\cfrac{n_1}{d_1+\cfrac{n_2}{d_2+\cdots}}=\cfrac{n_1}{d_1\cdot m+\cfrac{n_2}{d_2:m+\cfrac{n_3}{d_3\cdot m+\cfrac{n_4}{d_4:m+\cdots}}}},$$

$$-\cfrac{n_1}{d_1+\cfrac{n_2}{d_2+\cdots}}=\cfrac{n_1}{-d_1+\cfrac{n_2}{-d_2+\cfrac{n_3}{-d_3+\cfrac{n_4}{-d_4+\cdots}}}},$$

$$\cfrac{n}{d-\cfrac{n'}{d'-\cfrac{n''}{d''-\cfrac{n'''}{d'''-\cdots}}}}=\cfrac{n}{d+\cfrac{n'}{-d'+\cfrac{n''}{d''+\cfrac{n'''}{-d'''+\cdots}}}}.$$

9. Write all the primary and secondary convergents of $\frac{355}{113}$ in order; and find the fraction that differs least from it of all those whose denominators $\ngtr 100$. [§ 6, th. 6 cr. 2

10. Find all the fractions $\frac{\text{N}}{\text{D}}$ such that $\text{D}\ngtr 50$, $\frac{\text{N}}{\text{D}}\sim\frac{355}{113}<\frac{1}{50\cdot\text{D}}$.

11. Prove that $\left\{\begin{matrix}\text{N}_k\\ \text{D}_k\end{matrix}\right.=(n_k+d_{k-1}\cdot d_k)\left\{\begin{matrix}\text{N}_{k-2}\\ \text{D}_{k-2}\end{matrix}\right.+n_{k-1}\cdot d_k\left\{\begin{matrix}\text{N}_{k-3}\\ \text{D}_{k-3}\end{matrix}\right.$.

12. The continued fraction $\cfrac{1}{1\,\cfrac{1}{1\,\cfrac{1}{1\cdots}}}$ occurs in botany, zoölogy, and astronomy.

How does this fraction differ from $\sqrt{\frac{5}{4}}$?

Find twelve convergents, and prove that:

$$\begin{aligned}\text{N}_k&=2\cdot\text{N}_{k-2}+1\cdot\text{N}_{k-3}=\cdots\\ &=\text{N}_r\cdot\text{N}_{k-r+1}+\text{N}_{r-1}\cdot\text{N}_{k-r}\\ &=\text{N}_r\cdot\text{N}_{k-r+2}-\text{N}_{r-2}\cdot\text{N}_{k-r}\\ &=\tfrac{1}{2}(\text{N}_r\cdot\text{N}_{k-r+3}+\text{N}_{r-3}\cdot\text{N}_{k-r}),\quad\text{wherein } \tfrac{1}{2}\equiv\frac{1}{\text{N}_3}\\ &=\cdots\quad\cdots\quad\cdots\quad\cdots;\end{aligned}$$

that $\therefore$ N_r measures N_{2r}, N_{3r}, $\cdots$;
and that $\text{N}_r\cdot\text{N}_{s+t}-\text{N}_{r+s}\cdot\text{N}_t=(-1)^t\text{N}_{r-t}\cdot\text{N}_s$ if $r>t$.

13. Convert the series $a_0+a_0a_1x+a_0a_1a_2x^2+\cdots\pm a_0\cdots a_nx^n$ into the continued fraction, and find its first four convergents,

$$\cfrac{a_0}{1-\cfrac{a_1x}{a_1x+1-\cfrac{a_2x}{a_2x+1-\cdots\cfrac{a_nx}{a_nx+1}}}}.$$ [§ 2

VII. INCOMMENSURABLES, LIMITS, INFINITESIMALS, AND DERIVATIVES.

§ 1. VARIABLES AND CONSTANTS.—CONTINUITY.

WHEN the conditions of an investigation are such that one number takes a series of different values, that number is a *variable;* but a number that keeps its one value unchanged, is a *constant;* and the particular values that may be given to variables are constants. The remainder got by subtracting one value from the next is the *increment* of the value first named.

When one variable is a function of another, the first is a *dependent variable*, and the other is an *independent variable.* From the fixed values of the constants and the values that may be given to the independent variables, the corresponding values of the dependent variables, or functions, may be computed.

E.g., while a sum of money remains at interest, the principal and rate are constants, but the time and accrued interest are variables, of which either may be taken as the independent variable, and the other is dependent upon it; for when the principal, rate, and $\left\{\begin{matrix}\text{time}\\\text{interest}\end{matrix}\right.$ are given, the $\left\{\begin{matrix}\text{interest}\\\text{time}\end{matrix}\right.$ is thereby determined; and to different values given to $\left\{\begin{matrix}\text{time}\\\text{interest}\end{matrix}\right.$ different values of $\left\{\begin{matrix}\text{interest}\\\text{time}\end{matrix}\right.$ correspond.

So, the radius, circumference, and area of a circle are all functions of each other, and all grow together if the circle increases; but the ratio of the circumference to the radius is constant, and so is the ratio of the area to the square of either of them.

When the variable, in passing from one value to another, passes through every intermediate value in order, then the variable is *continuous;* otherwise it is *discontinuous.*

E.g., time is a continuous quantity, ever increasing by a steady growth, and the time of day, expressed in hours and parts of an hour, is a continuous number; but if even hours only be counted and the fractions rejected, the number is discontinuous.

If a function of a continuous variable remain real and finite as long as the variable is real and finite, if it can take but one value, or a limited number of values, for any one value of the variable, and if, in passing from one value to another, it passes through every intermediate value — such a function is a *continuous function;* otherwise it is *discontinuous.* It is implied that for any small increment of the variable the increment of the function is also small, and that to the variable an increment can always be given so small that the increment of the function shall be smaller than any assigned number.

E.g., interest is earned continuously, and may be computed for a year, a day, a minute, a second, a millionth part of a second, or any other fraction of a second, however small; interest is, therefore, a continuous function of the time. But in ordinary business fractions of a day are neglected, and interest, having definite sensible increments, is a discontinuous function of the time.

So, with a falling body, the force of gravity is constant, but the time, velocity acquired, and distance fallen are variables; and the velocity and distance are continuous functions of the time.

So, the area of a regular polygon inscribed in a given circle is a function of the number of sides, and varies with the number of sides; but neither the number of sides nor the area is a continuous number. For while there may be regular polygons with 3, 4, 5, ⋯ or any integral number of sides, it is absurd to speak of such a polygon of $3\frac{1}{2}$ sides, $4\frac{2}{3}$ sides, and so on.

So, the approximate value of the fraction $\frac{1}{3}$, expressed by the decimals .3, .33, .333, ⋯ is a function of the number of 3's employed, but that number is discontinous and so is the value.

So, the convergents of a continued fraction are functions of the partial numerators and denominators, but not continuous.

So, in the equation $4x^2 - 9y^2 = 36$, $y = \pm\frac{1}{3}\sqrt{(4x^2 - 36)}$, and for all values of $x < {}^-3$ and $> {}^+3$, y is a continuous function, but for all values of x from ${}^-3$ to ${}^+3$, y is discontinuous.

So, if $y = 1 : x$, y is a continuous function for all values of x except $x = 0$, where y leaps from a very large negative to a very large positive value.

§ 2. INCOMMENSURABLES.

If, in any operation upon numbers, the result cannot be expressed as a commensurable number, either an integer or a fraction, but commensurable numbers can be found both greater and less than the true result that approach indefinitely near to it and to each other, such result is an *incommensurable* number.

E.g., the square root of 2 is an incommensurable.

(*a*) *It is not an integer.*

For $(0)^2 = 0,\ (\pm 1)^2 = 1,\ (\pm 2)^2 = 4,$

and $(\pm 3)^2,\ (\pm 4)^2, \cdots$ each > 2.

(*b*) *It is not a simple fraction.*

For if possible, let $\sqrt{2} = \frac{m}{n}$, a simple fraction in its lowest terms;

then $2 = \frac{m^2}{n^2}$ a simple fraction in its lowest terms, [III. th.5, cr.3

i.e., an integer is equal to an irreducible fraction, which is absurd;

$\therefore \sqrt{2}$ is not a simple fraction. Q. E. D.

(*c*) *Commensurables, both greater and less than* $\sqrt{2}$, *can be found, that shall differ from it by less than any assigned number, however small.*

For $\because (\pm 1)^2 = 1$, and $(\pm 2)^2 = 4$,

$\therefore \pm 1 \lessdot \sqrt{2}$, and $\pm 2 \gtrdot \sqrt{2}$,

and each of them $\sim \sqrt{2} < 1$.

So $\because (\pm 1.4)^2 = 1.96$, and $(\pm 1.5)^2 = 2.25$,

$\therefore \pm 1.4 \lessdot \sqrt{2}$, and $\pm 1.5 \gtrdot \sqrt{2}$,

and each of them $\sim \sqrt{2} < .1$.

So $\because (\pm 1.41)^2 = 1.9881$, and $(\pm 1.42)^2 = 2.0164$,

$\therefore \pm 1.41 \lessdot \sqrt{2}$, and $\pm 1.42 \gtrdot \sqrt{2}$,

and each of them $\sim \sqrt{2} < .01$.

So indefinitely, however small the difference assigned.

So the square roots of 4, 9, 16, 25, 36, 49, 64, and 81 are commensurables; but of all other integers, and of most fractions, lying between 1 and 100 they are incommensurables; and so of other roots.

So the logarithms, to the base 10, of 10, 100, and 1000 are commensurable; but of all other integers between 1 and 10000 they are incommensurables.

Incommensurable numbers often represent the attempt to express the numerical measure of a quantity in terms of a unit that has no common measure with it. If expressed in terms of some other unit, the number might be commensurable.

E.g., the diagonal of a square is incommensurable with its side; but in terms of the half diagonal, or any other exact part of itself, say $\frac{7}{11}$ths, $\frac{28}{43}$ds, $\frac{53}{48}$ths, $\cdots$, it is commensurable.

So, time may be expressed in days, in lunar months, or in years; but it is very unlikely that a given length of time, exactly expressed in any one of these units, would be commensurable in either of the others.

So, if two distances, A —— B and C —— D, be taken at random, the chances are few that AB is a measure of CD, or that they even have a common measure. If they seem to have one, it is probably because most measurements are inexact, and only rough approximations are used instead of the true numbers, which are commonly incommensurable.

The words addition, subtraction, multiplication, division, and involution to commensurable powers, were defined in I. §§ 6–11; and those definitions were made so broad as to cover all kinds of numbers. The axioms laid down in II. § 3 also apply to all numbers.

Incommensurable powers and logarithms are defined in VIII. § 4, IX. § 1.

The combinatory properties of commensurable numbers were proved in II. §§ 4, 6, 7; the same properties are proved for incommensurables in VII. § 7.

§ 3. LIMITS.

WHEN a variable takes successive values that approach nearer and nearer to a given constant, so that the difference between the variable and the constant is very small, and may become and remain smaller than any number named or conceived of, then the constant is the *limit* of the variable; and this definition applies whether the variable be always greater, or always less, or sometimes greater and sometimes less, than the constant.

E.g., 1, 1.4, 1.41, 1.414, 1.4142, ⋯
are successive approximations to the true value of $\sqrt{2}$,
and if the series be extended, a succession of terms may be found whose differences from $\sqrt{2}$ are smaller than any assigned number, and steadily grow smaller and smaller as the series goes on, but which terms are each less than $\sqrt{2}$.

So 2, 1.5, 1.42, 1.415, 1.4143, ⋯
are each greater than $\sqrt{2}$, but approach it nearer and nearer without end;
$\therefore$ while $\sqrt{2}$ can never be exactly expressed in decimals, it is the limit to which both the series approach.

So as shown under continued fractions, $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}$
and the successive convergents,
$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169}, \cdots$
are alternately greater and less than $\sqrt{2}$, the true value, but approach it nearer and nearer as a limit.

So 1 is the limit $\frac{n \pm 1}{n}$ when n increases without bounds.

For $\because \frac{n \pm 1}{n} = 1 \pm \frac{1}{n}$, and $\frac{1}{n}$, $= 1 \pm \frac{1}{n} \sim 1$, $\doteq 0$ when $n \doteq \infty$,

wherein "$\doteq 0$" $\equiv$ grows smaller and smaller and approaches 0 as its limit,

and "$\doteq \infty$" $\equiv$ grows larger and larger without bounds;

$\therefore \lim \frac{n \pm 1}{n} = 1$, when $n \doteq \infty$. Q.E.D.

So $\frac{1}{3}$ is the limit of the series .3, .33, .333, .3333, ⋯.

So if from the series $\frac{1}{2}$, $(\frac{1}{2})^2$, $(\frac{1}{2})^3$, $(\frac{1}{2})^4$, ⋯ $(\frac{1}{2})^n$ a new series of sums be formed by taking

$$s_1 = \frac{1}{2},\ s_2 = \frac{1}{2} + (\frac{1}{2})^2 = \frac{3}{4},\ s_3 = \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 = \frac{7}{8},\ \cdots;$$

then the limit of this series, when $n \doteq \infty$, is 1.

For $s_1 \sim 1 = \frac{1}{2}$, $s_2 \sim 1 = \frac{1}{4} = (\frac{1}{2})^2$, $s_3 \sim 1 = \frac{1}{8} = (\frac{1}{2})^3$, ⋯,

and $s_n \sim 1 = (\frac{1}{2})^n \doteq 0$, when $n \doteq \infty$. Q. E. D.

So, if a regular polygon be inscribed in a circle, and another be circumscribed about it, and if the number of their sides be doubled again and again, the area of the circle is the limit of the areas of both the polygons, and the circumference of the circle is the limit of their perimeters. The two areas approach nearer and nearer to the area of the circle and to each other; but one is always a little greater and the other a little less than the circle; and so of the perimeters.

So, the surface and volume of a cone are the limits respectively of the surface and volume of an inscribed, and of a circumscribed, pyramid; the surface and volume of a cylinder are the limits of the surface and volume of an inscribed, and a circumscribed, prism, and so on.

In these examples, as in all others, the constants $\sqrt{2}$, 1, $\frac{1}{3}$, ⋯ are limits, not simply because the successive values of the variables approach nearer and nearer to them, for they approach nearer and nearer to many other numbers not their limits.

E.g., the series 1, 1.4, 1.41, 1.414, 1.4142, ⋯ approaches nearer and nearer to 10 000, which is not its limit at all.

So, the area of the inscribed polygon approaches nearer and nearer to the area of the circumscribed square, not its limit.

The constants are limits because, as the series is extended, some one of its terms, and all the terms that follow it, will differ from the constant by a number smaller than any assigned number, be that number never so small; and further, because, however far the series is extended, there is no point beyond which its successive terms are each of them equal to the limit.

§ 4. INFINITESIMALS AND INFINITES.

A VARIABLE is *infinitesimal* if it can take values smaller than any assignable magnitude, *infinite* if it can take values larger than any assignable magnitude, *finite* if neither infinite nor infinitesimal. All constants except 0 are finite. Strictly, the word "infinitesimal" applies only to that part of the series of values of the variable which are smaller than any number that can be named or conceived of, and "infinite" to that part of the series of values which are larger than can be named or conceived of.

Manifestly, the difference between a variable and its limit is an infinitesimal.

The reader must carefully note the distinction between an infinitesimal and absolute nothing. The latter comes from subtracting any number from itself; the former from dividing any number into small parts and then continually subdividing one of these parts. An infinitesimal always has some magnitude, but absolute nothing means the total absence of anything to measure.

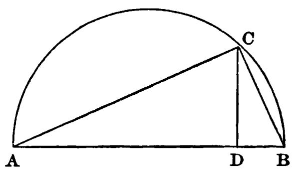
So, between the infinity of mathematics and the absolute infinity of space and duration, there is the same impassable gulf. Absolute infinity means that boundlessness to which nothing can be added, and from which nothing can be taken away, and there are no means by which it can be measured; but mathematical infinity is simply "a number larger than can be named or conceived of," and one such infinity may be larger than another, or any number of times another.

The essential properties of infinitesimals and infinites, upon which the mathematician rests, are: that, while following the law by which successive values are determined, the one may be made smaller and smaller, and the other larger and larger, at pleasure.

In comparing two or more $\left\{ \begin{array}{l} \text{infinitesimals} \\ \text{infinites} \end{array} \right.$ any one of them may be assumed at pleasure as the *base* of the system, and if the *limiting ratio* (limit of ratio) of any other to the base be a finite number, not 0, it is of the *same order as the base*. If this ratio be not a finite number, other than 0, it is of *different order*. If it be 0, it is an *infinitesimal as to the base*. If it be ∞, it is an

infinite as to the base. If the limiting ratio of an $\left\{\begin{matrix}\text{infinitesimal}\\ \text{infinite}\end{matrix}\right.$ to the nth power of the base (whatever n) be a finite number, not 0, the $\left\{\begin{matrix}\text{infinitesimal}\\ \text{infinite}\end{matrix}\right.$ is of the nth *order* as to the base, and two $\left\{\begin{matrix}\text{infinitesimals}\\ \text{infinites}\end{matrix}\right.$ are of the same order if they have finite limiting ratios to the same power of the base.

E.g., if upon any straight line AB a semicircle be described, and from C, any point of the circumference, CD be drawn perpendicular to AB, and AC and CB be joined,



then ABC and CBD are similar right triangles,

and AB : BC = BC : DB.

Let C move towards B,

then AB is constant and BC and DB are variables.

Let C approach indefinitely near to B,

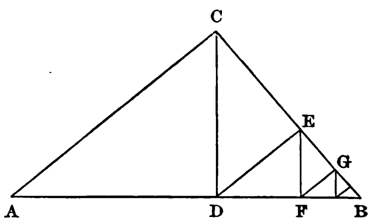
then BC is an infinitesimal of the first order,

and DB of the second order, as to BC.

For $\because$ AB · DB = BC^2, [above

$\therefore$ lim (BC^2 : DB) = AB a finite length. Q.E.D. [df.

So, if in the triangle ABC, right-angled at C, perpendiculars be let fall from C on AB at D, from D on BC at E, from E on DB at F, from F on BE at G, and so on;



then the triangles ABC, CBD, DBE, EBF, FBG, ··· are all similar,

and BC : AB = DB : BC = BE : DB = FB : BE = BG : FB = ···.

Conceive the point C to move towards B, and to approach indefinitely near to it, then the ratios grow smaller and smaller, and finally become infinitesimals, and the lengths DB, BE, FB, BG, ··· are infinitesimals of the 1st, 2d, 3d, 4th, ··· orders as to the constant length AB.

§ 5. DERIVATIVES.

If to a variable a small increment be given, and if the corresponding increment of a function of the variable be determined, then the limit of the ratio of the increment of the function to the increment of the variable, when the increment of the variable is taken indefinitely small, is the *derivative of the function as to the variable.*

E.g., let a square pyramid be cut by planes parallel to the base; the sections are squares, and they grow larger as the cutting planes recede from the vertex.

Take the sides of two squares 6 inches and 7 inches;

then $\quad (7^2-6^2):(7-6)=13:1=13.$

Take the sides of two square 6 inches and 6.1 inches

then $\quad (6.1^2-6^2):(6.1-6)=1.21:.1=12.1.$

Take the sides of two squares 6 inches and 6.01 inches,

then $\quad (6.01^2-6^2):(6.01-6)=.1201:.01=12.01.$

Take the sides of two squares 6 inches and 6.001 inches,

then $\quad (6.001^2-6^2):(6.001-6)=.012001:.001=12.001;$

It thus appears that as the difference of sides grows smaller,

$1, .1, .01, .001, \cdots$ towards 0,

so also the difference of areas grows smaller indefinitely,

$13, 1.21, .1201, .012001, \cdots$ towards 0,

but that the ratio of these differences, though growing smaller, has 12 and not 0 for its limit.

$13, 12.1, 12.01, 12.001, \cdots$ towards 12;

i.e., just as the side of the square reaches and passes 6 inches in its growth, at that instant the area is growing 12 times as fast as the side; as it reaches and passes 7 inches, 14 times as fast; as it reaches and passes 8 inches, 16 times as fast, and so on; and, in general, as it reaches and passes x inches, $2x$ times as fast.

When two variables grow smaller and smaller together, their ratio does not necessarily, nor generally, become infinitesimal.

E.g., if a be any number, however small, and mb, nb be smaller than a,

then $\quad mb:nb=m:n$, whatever m and n may be.

So let $x = 1, \frac{1}{20}, \frac{1}{400}, \frac{1}{8000}, \frac{1}{160000}, \cdots$ towards 0,

and $y = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ towards 0;

then $x : y = 1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \cdots$ towards 0,

and $y : x = 1, 10, 100, 1000, 10000, \cdots$ towards ∞.

So let $x = \frac{2}{1 \cdot 2}, \frac{2}{2 \cdot 3}, \frac{2}{3 \cdot 4}, \frac{2}{4 \cdot 5}, \frac{2}{5 \cdot 6}, \frac{2}{6 \cdot 7}, \frac{2}{7 \cdot 8}, \cdots$ towards 0,

and $y = 1, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \frac{1}{5^2}, \frac{1}{6^2}, \frac{1}{7^2}, \cdots$ towards 0;

then $x : y = 1, \frac{4}{3}, \frac{9}{6}, \frac{16}{10}, \frac{25}{15}, \frac{36}{21}, \frac{49}{28}, \cdots$ towards 2.

If y be a function of x, then the phrase "derivative of y as to x," is written $\mathrm{D}_x y$, wherein D stands for "derivative of," and the subscript x for "as to x." This phrase is read more briefly, "the x derivative of y."

So, $\mathrm{D}_y x \equiv$ the derivative of x as to y, or the y derivative of x.

Manifestly, $\mathrm{D}_y x$ is the reciprocal of $\mathrm{D}_x y$, *i.e.*, $\mathrm{D}_x y \cdot \mathrm{D}_y x = 1$,

for $\because \frac{\text{inc. } x}{\text{inc. } y} \cdot \frac{\text{inc. } y}{\text{inc. } x} = 1$, however nearly the fractions have come to their limits $\mathrm{D}_y x$, $\mathrm{D}_x y$,

$\therefore \mathrm{D}_y x \cdot \mathrm{D}_x y = 1.$

Under the general heading of this chapter the reader will find three classes of problems, and the theorems that follow lay the foundation for rules for their solution:

1. Those which involve the limits of variables.
2. Those which involve the ratios of two infinitesimals.
3. Those which involve the sums of an infinite number of infinitesimals.

To the first class belong the various examples given under the head of incommensurables and limits; to the second class belong those under the head of derivatives; and to the third, the computation of areas and volumes, with other like problems. The process last named is called *integration*.

§ 6. FIRST PRINCIPLES.

THEOR. I. *If two variables, the one increasing and the other decreasing, approach each other so that their difference becomes infinitesimal, they have a common limit that lies between them:*

1. Each of them has some limit.

For, if either had no limit, they would pass each other.

2. They have the same limit.

For, if they had different limits they could come no nearer together than their limits. [§ 3 df. lim.

3. This common limit lies between the two variables.

For it is greater than the less, and less than the greater of them.

COR. *If two constants always lie between two such variables, they are equal to the common limit and to each other.*

For, if possible, let one of them be greater than the limit;
then the greater variable can get no nearer the limit than this constant, which is absurd. [§ 3 df. lim.
So, neither of the constants can be less than the limit.
∴ they are equal to the limit, and to each other. Q.E.D.

THEOR. 2. *The product of a finite number into an infinitesimal is an infinitesimal, and of the same order.*

Let n be any finite number, and a an infinitesimal;
then will $n \cdot a$ be an infinitesimal, and of the same order as a.

1. $n \cdot a$ is an infinitesimal.

For, take β any finite number however small, and $a < \beta : n$;
then $n \cdot a < \beta$, and is inf'l. Q.E.D. [§ 4 df. infl. II. ax. 16

2. $n \cdot a$ is an infinitesimal of the same order as a.

For $n \cdot a : a, = n$, is finite. Q.E.D. [df. infl. of same ord.

COR. 1. *The sum of a finite number,* n, *of infinitesimals is inf'l.*

For their sum $< n$ times the largest of them, [II. ax. 12
and ∵ that product is infinitesimal, [th.
∴ the sum is infinitesimal. Q.E.D.

COR. 2. *If there be any finite number of commensurable variables,* $x', y', z', \cdots$, *and as many more,* $x'', y'', z'', \cdots$, *such that*

$$x' \sim x'' \doteq 0,\ y' \sim y'' \doteq 0,\ z' \sim z'' \doteq 0, \cdots;$$

then will $\overline{x' + y' + z' + \cdots} \sim \overline{x'' + y'' + z'' + \cdots} \doteq 0,$

and $\overline{x' \cdot y' \cdot z' \cdot \cdots} \sim \overline{x'' \cdot y'' \cdot z'' + \cdots} \doteq 0.$

§ 7. PRIMARY OPERATIONS ON INCOMMENSURABLES.

THEOR. 3. *The addition of incommensurables is commutative and associative.*

For, let a, b, c, $\cdots$ be any incommensurables,

$$\text{and let}\left\{\begin{array}{l} x',\ x'' \\ y',\ y'' \text{ be commensurable} \\ z',\ z'' \text{ variables such that} \\ \cdots,\ \cdots \end{array}\right\}\begin{array}{l} x' < a < x'', \\ y' < b < y'', \\ z' < c < z'', \\ \cdots\quad \cdots\quad \cdots, \end{array} \qquad [\S\ 2 \text{ df.}$$

and such that a is the limit of x' and x'', b of y' and y'', c of z' and z'', $\cdots$;

then $\because\ x' + y' + z' + \cdots < a + b + c + \cdots < x'' + y'' + z'' + \cdots,$ [II. ax. 12

and $z' + y' + x' + \cdots < c + b + a + \cdots < z'' + y'' + x'' + \cdots,$

and $\because\ x' + y' + z' + \cdots, = z' + y' + x' + \cdots,$ [II. th. 1

$\doteq x'' + y'' + z'' + \cdots, = z'' + y'' + x'' + \cdots,$ [th. 2 cr. 2

$\therefore$ the constant sums $a + b + c + \cdots$, $c + b + a + \cdots$, lying between these two variable sums, are equal. [th. 1

So for any other order or grouping of the elements in the sum of a, b, c, $\cdots$. Q. E. D.

THEOR. 4. *The multiplication of incommensurables is commutative and associative.*

For, let a, b, c, $\cdots$ be any incommensurables,

$$\text{and let}\left\{\begin{array}{l} x',\ x'' \\ y',\ y'' \text{ be commensurable} \\ z',\ z'' \text{ variables such that} \\ \cdots,\ \cdots \end{array}\right\}\begin{array}{l} x' \leqslant a \leqslant x'', \\ y' \leqslant b \leqslant y'', \\ z' \leqslant c \leqslant z'', \\ \cdots\qquad \cdots, \end{array} \qquad [\S\ 2 \text{ df.}$$

and such that a is the limit of x' and x'', b of y' and y'', c of z' and z'', $\cdots$;

then $\because\ x' \cdot y' \cdot z' \cdots \leqslant a \cdot b \cdot c \cdots \leqslant x'' \cdot y'' \cdot z'' \cdots,$ [II. ax. 19

and $z' \cdot y' \cdot x' \cdots \leqslant c \cdot b \cdot a \cdots \leqslant z'' \cdot y'' \cdot x'' \cdots,$

and $\because\ x' \cdot y' \cdot z' \cdots, = z' \cdot y' \cdot x' \cdots,$ [II. th. 3

$\doteq x'' \cdot y'' \cdot z'' \cdots, = z'' \cdot y'' \cdot x'' \cdots,$ [th. 2 cr. 2

$\therefore$ the constant products $a \cdot b \cdot c \cdots$, $c \cdot b \cdot a \cdots$, lying between these two variable products, are equal. [th. 1 cr. 1

So for any other order or grouping of the elements in the product of a, b, c, $\cdots$. Q. E. D.

THEOR. 5. *The multiplication of incommensurables is distributive as to addition.*

The proof is identical with that of [II. th. 4.].

§ 8. GENERAL PROPERTIES OF LIMITS.

THEOR. 6. *If two variables be always equal, and if one of them approach a limit, the other approaches the same limit.*

For, let x, y be two variables, always equal, and a the limit of x;

then $\because$ $x \sim a \doteq 0$, [§ 3 df.

and $\because$ $y = x$ always, [hyp.

$\therefore$ $y \sim a \doteq 0$; [I. § 5 df.

i.e., a is the limit of y. Q. E. D.

COR. *If, while approaching their limits, two variables be always equal, their limits are equal.*

NOTE. Another and independent demonstration of this corollary is as follows:

Let x, y be two variables, always equal, and a, b their limits, then will $a = b$.

For if not, let $a \sim b = \delta$,

then $\because$ a, b are both constants,

$\therefore$ δ is a constant, however small it may be.

Take x, y such that $x \sim a < \frac{1}{2}\delta$, and $y \sim b < \frac{1}{2}\delta$, [df.

then $\because$ $x = y$ always, [hyp.

$\therefore$ $a \sim b < \delta$, [II. ax. 12

which is contrary to the supposition that $a \sim b = \delta$;

$\therefore$ that supposition fails,

and it is only left that $a = b$. Q. E. D.

THEOR. 7. *If there be any finite number of variables having limits, the sum of their limits is the limit of their sum.*

Let x, y, z, $\cdots$ be any finite number of variables, and a, b, c, $\cdots$ their limits; then is the sum $a + b + c + \cdots$ the limit of the sum $x + y + z + \cdots$.

For $\because$ $x = a + \alpha$, $y = b + \beta$, $z = c + \gamma$, $\cdots$,

wherein α, β, γ, $\cdots$ may be positive or negative, but each of them $\doteq 0$,

$\therefore\ x+y+z+\cdots=(a+\alpha)+(b+\beta)+(c+\gamma)+\cdots$ [II. ax. 2

$=(a+b+c+\cdots)+(\alpha+\beta+\gamma+\cdots)$; [th. 3

and $\because\ \alpha+\beta+\gamma+\cdots\doteq 0$, [th. 2

$\therefore\ x+y+z+\cdots\doteq a+b+c+\cdots$ as its limit. Q.E.D. [§ 2 df.

NOTE. When the number of terms is infinite this theorem does not always apply.

E.g., if a, a finite number, be divided into x parts,

then $\lim\left(\frac{a}{x}+\frac{a}{x}+\frac{a}{x}+\cdots \text{ to } x \text{ terms}\right)$ is a when $x\doteq\infty$ and $\frac{a}{x}\doteq 0$.

THEOR. 8. *If there be any finite number of variables having limits, the product of their limits is the limit of their product.*

Let $x, y, z, \cdots$ be any variables, $a, b, c, \cdots$ their limits; then is the product $a\cdot b\cdot c\cdots$ the limit of the product $x\cdot y\cdot z\cdots$.

For $\because\ x=a+\alpha,\quad y=b+\beta,\quad z=c+\gamma,\ \cdots$,

wherein $\alpha, \beta, \gamma, \cdots$ may be positive or negative, but each of them $\doteq 0$,

$\therefore\ x\cdot y\cdot z\cdots=(a+\alpha)\cdot(b+\beta)\cdot(c+\gamma)\cdots$ [II. ax. 4

$=a\cdot b\cdot c\cdots+$, a finite number of terms, each of which has one or more of the factors $\alpha, \beta, \gamma, \cdots$, and is therefore an infinitesimal; [ths. 5, 3

and $\because$ the sum of finite multiples of $\alpha, \beta, \gamma, \cdots\doteq 0$, [th. 2

$\therefore\ x\cdot y\cdot z\cdots\doteq a\cdot b\cdot c\cdots$ as its limit. Q.E.D. [df.

COR. 1. *If there be two variables, the quotient of their limits is the limit of their quotient.*

Let x, y be any two variables, and a, b their limits; then is the quotient $a:b$ the limit of the quotient $x:y$.

For let $x=y\cdot q$, wherein q is the quotient of x by y,

then $\because\ a=b\cdot\lim q$, [th.

$\therefore\ a:b=\lim q$. Q.E.D.

COR. 2. *Any finite integral power of the limit of a variable is the limit of the like power of the variable.*

Let x be any variable, a its limit, n any integer, then $a^n=\lim x^n$.

(a) n *positive:* a case of multiplication.

(b) n *negative:* a case of division.

NOTE. When the exponent is infinite, this corollary does not always apply.

E.g., $\left(1+\frac{1}{x}\right)^x$, when $x \doteq \infty$, is not 1 but $2.718+\cdots$, [

COR. 3. *If there be two variables* x, y *and two others* x′, y′, *such that* $\lim(\mathrm{x}:\mathrm{x}')=1$ *and* $\lim(\mathrm{y}:\mathrm{y}')=1$,

then $\lim(\mathrm{x}:\mathrm{y})=\lim(\mathrm{x}':\mathrm{y}')$.

THEOR. 9. *If there be two variables* x, y *whose limiting ratio is a finite number, not* 0, *and if there be added to them any numbers* α, β, *infinitesimal as to* x, y, *then is the limiting ratio of* x, y *not changed.*

For $\because \dfrac{x+\alpha}{y+\beta}=\dfrac{x}{y}\cdot\dfrac{1+(\alpha:x)}{1+(\beta:y)}$,

$$\therefore \lim\frac{x+\alpha}{y+\beta}=\lim\frac{x}{y}\cdot\lim\frac{1+(\alpha:x)}{1+(\beta:y)}=\lim\frac{x}{y}\cdot\frac{1}{1}=\lim\frac{x}{y}.$$

Q. E. D.

COR. *If the difference,* δ, *of two variables* x, y *be infinitesimal as to either, their limiting ratio is* 1, *and conversely.*

THEOR. 10. *If* x, y, z, *be three variables of the same sense, such that* $\mathrm{x} \leqslant \mathrm{y} \leqslant \mathrm{z}$, *and such that* $\mathrm{x}:\mathrm{z}\doteq 1$, *then will* $\mathrm{x}:\mathrm{y}\doteq 1$, *and* $\mathrm{y}:\mathrm{z}\doteq 1$.

For $\because x \leqslant y \leqslant z$, [hyp.

$\therefore x:z \leqslant x:y \leqslant y:y$, *i.e.*, $\leqslant 1$, [II. ax. 18

and $x:z \leqslant y:z \leqslant z:z$, *i.e.*, $\leqslant 1$. [II. ax. 17

But $\because x:z \doteq 1$, [hyp.

$\therefore x:y, \geqslant x:z$ but $\leqslant 1, \doteq 1$,

and $y:z, \geqslant x:z$ but $\leqslant 1, \doteq 1$. Q. E. D.

THEOR. 11. *If* x, y *be two infinitesimals,* m, n *their orders as to any base* β, *and if* $\mathrm{m}>\mathrm{n}$, *then is* (x : y) *an infinitesimal of the* (m − n)*th order as to the base.*

For $\because \lim(x:\beta^m)=h$ and $\lim(y:\beta^n)=k$,

wherein h and k are finite numbers, [hyp.

$\therefore \dfrac{\lim(x:y)}{\lim\beta^{m-n}},=\dfrac{\lim(x:\beta^m)}{\lim(y:\beta^n)},=\dfrac{h}{k}$, a finite number;

$\therefore (x:y)$ is an infinitesimal of the $(m-n)$th order. Q. E. D.

COR. 1. *The product* $x \cdot y$ *is an infinitesimal of the* $(m+n)$*th order.*

COR. 2. $y : x$ *is an infinite of the* $(m-n)$*th order.*

COR. 3. *If* x, y *be infinites of the* m*th and* n*th orders, and if* $m > n$, *then:*

$x : y$ *is an infinite of the* $(m-n)$*th order,*

$x \cdot y$ *is an infinite of the* $(m+n)$*th order,*

$y : x$ *is an infinitesimal of the* $(m-n)$*th order.*

If there be two or more numbers not all equal, then any number which is greater than the least of them and less the greatest is a *mean*. The *average* of two or more numbers is the quotient of their sum by their number.

THEOR. 12. *If* x', x'', x''', $\cdots$ *be a set of variables, and* y', y'', y''', $\cdots$ *as many more, all positive or all negative, and such that* $\lim(x' : y') = 1$, $\lim(x'' : y'') = 1$, $\lim(x''' : y''') = 1$, $\cdots$; *and if the number of variables in each set increase without bounds, then the limits of the sums of the two sets, unless infinite, are equal.*

For $\because \lim(x' : y') = 1,\ \lim(x'' : y'') = 1,\ \lim(x''' : y''') = 1, \cdots,$ [hyp.

and $\because \dfrac{x'+x''+x'''+\cdots}{y'+y''+y'''+\cdots} = \text{mean}\left(\dfrac{x'}{y'}, \dfrac{x''}{y''}, \dfrac{x'''}{y'''}, \cdots\right),$

$$\therefore \lim \frac{x'+x''+x'''+\cdots}{y'+y''+y'''+\cdots} = 1,$$

$$\therefore \frac{\lim(x'+x''+x'''+\cdots)}{\lim(y'+y''+y'''+\cdots)} = 1,$$

$$\therefore \lim(x'+x''+x'''+\cdots) = \lim(y'+y''+y'''+\cdots). \quad \text{Q.E.D.}$$

COR. *If* $\lim x' : y' = m$, $\lim x'' : y'' = m$, $\cdots$, *and if* x', x'', $\cdots$ y', y'', $\cdots$ *be all positive or all negative, then*

$$\lim[(x'+x''+\cdots) : (y'+y''+\cdots)] = m.$$

§ 9. GENERAL PROPERTIES OF DERIVATIVES.

PROB. 1. TO FIND THE DERIVATIVE, AS TO ANY VARIABLE, OF A FUNCTION OF THAT VARIABLE.

In the function give the variable an increment; from the resulting expression subtract the function; divide the remainder by the increment of the variable, and get the limit of the quotient as that increment approaches zero.

E.g., to find the derivative of x^2:

Let $\mathrm{x} \equiv x^2$; substitute $x+h$ for x, and let $\mathrm{x}' \equiv (x+h)^2$;

then $\because \mathrm{x}' - \mathrm{x} = (x+h)^2 - x^2 = 2xh + h^2$,

$$\therefore \frac{\mathrm{x}'-\mathrm{x}}{h} = 2x + h,$$

$$\therefore \lim \frac{\mathrm{x}'-\mathrm{x}}{h} = 2x \quad \text{when } h \doteq 0,$$

i.e., $\mathrm{D}_x(x^2) = 2x$. Q.E.F.

So to find the derivative of x^3:

Let $\mathrm{x} \equiv x^3$; substitute $x+h$ for x, and let $\mathrm{x}' \equiv (x+h)^3$;

then $\because \mathrm{x}' - \mathrm{x} = (x+h)^3 - x^3 = 3x^2h + 3xh^2 + h^3$,

$$\therefore \frac{\mathrm{x}'-\mathrm{x}}{h} = 3x^2 + 3xh + h^2,$$

$$\therefore \lim \frac{\mathrm{x}'-\mathrm{x}}{h} = 3x^2 \quad \text{when } h \doteq 0,$$

i.e., $\mathrm{D}_x(x^3) = 3x^2$.

So to find the derivative of x^{-1}:

Let $\mathrm{x} \equiv x^{-1}$; substitute $x+h$ for x, and let $\mathrm{x}' \equiv (x+h)^{-1}$;

then $\because \mathrm{x}' - \mathrm{x} = (x+h)^{-1} - x^{-1}$

$$= \frac{1}{x+h} - \frac{1}{x} \qquad = \frac{-h}{x(x+h)},$$

$$\therefore \frac{\mathrm{x}'-\mathrm{x}}{h} = \frac{-1}{x(x+h)},$$

$$\therefore \lim \frac{\mathrm{x}'-\mathrm{x}}{h} = \frac{-1}{x^2} \quad \text{when } h \doteq 0,$$

i.e., $\mathrm{D}_x \frac{1}{x} = -\frac{1}{x^2}$. Q.E.F.

THEOR. 13. *The derivative, as to any variable, of the sum of two or more functions of that variable, is the sum of their derivatives.*

Let $\text{U}, \text{V} \cdots$ be any functions of a variable x, and X their sum; then will $\text{D}_x\text{X} = \text{D}_x\text{U} + \text{D}_x\text{V} + \cdots$

For let x take any infinitesimal increment h, and let x' stand for the new value of x, X' for the corresponding value of X, U' for that of U, V' for that of V, $\cdots$, so that $x' \equiv x + h$, $\text{X}' \equiv \text{X} + \text{inc}\,\text{X}$, $\text{U}' \equiv \text{U} + \text{inc}\,\text{U}$, $\text{V}' \equiv \text{V} + \text{inc}\,\text{V}$,

then $\because$ $\text{X} = \text{U} + \text{V} + \cdots$ always, [hyp.

$\therefore$ $\text{X}' = \text{U}' + \text{V}' + \cdots$,

$\therefore$ $\text{X}' - \text{X} = \overline{\text{U}' - \text{U}} + \overline{\text{V}' - \text{V}} + \cdots$, [II. ax. 3

i.e., $\text{inc}\,\text{X} = \text{inc}\,\text{U} + \text{inc}\,\text{V} + \cdots$,

$\therefore$ $\dfrac{\text{inc}\,\text{X}}{h} = \dfrac{\text{inc}\,\text{U}}{h} + \dfrac{\text{inc}\,\text{V}}{h} + \cdots$, [II. ax. 5

$\therefore$ $\lim \dfrac{\text{inc}\,\text{X}}{h} = \lim \dfrac{\text{inc}\,\text{U}}{h} + \lim \dfrac{\text{inc}\,\text{V}}{h} + \cdots$ when $h \doteq 0$, [th. 7

i.e., $\text{D}_x\text{X}, = \text{D}_x(\text{U} + \text{V} + \cdots)$,

$= \text{D}_x\text{U} + \text{D}_x\text{V} + \cdots$. Q.E.D.

THEOR. 14. *The derivative, as to any variable, of the product of two or more functions of that variable, is the sum of the products of the derivatives of the several factors each multiplied by all the other factors.*

Let $\text{U}, \text{V}, \text{W}, \cdots$ be any functions of a variable x, and X their product;

then will $\text{D}_x\text{X} = \text{V}\cdot\text{W}\cdots\text{D}_x\text{U} + \text{U}\cdot\text{W}\cdots\text{D}_x\text{V} + \text{U}\cdot\text{V}\cdots\text{D}_x\text{W} + \cdots$.

For let x take any infinitesimal increment h, and x' be the new value of x, so that $x' \equiv x + h$, $\text{X}' \equiv \text{X} + \text{inc}\,\text{X}, \cdots$,

then $\because$ $\text{X} = \text{U}\cdot\text{V}\cdot\text{W}\cdots$ always, [hyp.

$\therefore$ $\text{X}' = \text{U}'\cdot\text{V}'\cdot\text{W}'\cdots$,

i.e., $\text{X} + \text{inc}\,\text{X} = (\text{U} + \text{inc}\,\text{U})\cdot(\text{V} + \text{inc}\,\text{V})\cdot(\text{W} + \text{inc}\,\text{W})\cdots$

$= \text{U}\cdot\text{V}\cdot\text{W}\cdots + \text{V}\cdot\text{W}\cdots\text{inc}\,\text{U} + \text{U}\cdot\text{W}\cdots\text{inc}\,\text{V} + \cdots$

$+$ terms with two or more infin'l factors,

$$\therefore\ \text{inc}\,\mathrm{X} = \mathrm{V}\cdot\mathrm{W}\cdots\text{inc}\,\mathrm{U} + \mathrm{U}\cdot\mathrm{W}\cdots\text{inc}\,\mathrm{V} + \mathrm{U}\cdot\mathrm{V}\cdots\text{inc}\,\mathrm{W} + \cdots$$
$$+ \text{terms with two or more infin'l factors,}$$

$$\therefore\ \frac{\text{inc}\,\mathrm{X}}{h} = \mathrm{V}\cdot\mathrm{W}\cdots\frac{\text{inc}\,\mathrm{U}}{h} + \mathrm{U}\cdot\mathrm{W}\cdots\frac{\text{inc}\,\mathrm{V}}{h} + \mathrm{U}\cdot\mathrm{V}\cdots\frac{\text{inc}\,\mathrm{W}}{h} + \cdots$$
$$+ \text{terms with one or more infin'l factors,}$$

$$\therefore\ \mathrm{D}_x\mathrm{X},\ = \mathrm{D}_x(\mathrm{U}\cdot\mathrm{V}\cdot\mathrm{W}\cdots),$$
$$= \mathrm{V}\cdot\mathrm{W}\cdots\mathrm{D}_x\mathrm{U} + \mathrm{U}\cdot\mathrm{W}\cdots\mathrm{D}_x\mathrm{V} + \mathrm{U}\cdot\mathrm{V}\cdots\mathrm{D}_x\mathrm{W} + \cdots$$
$$+ \text{terms that vanish.} \qquad \text{Q.E.D.} \qquad [\text{th. } 7$$

COR. *In particular, the derivative, as to any variable, of the product of two functions of that variable, is the sum of the products of the derivatives of the two functions each multiplied by the other function.*

i.e., $\mathrm{D}_x(\mathrm{U}\cdot\mathrm{V}) = \mathrm{U}\cdot\mathrm{D}_x\mathrm{V} + \mathrm{V}\cdot\mathrm{D}_x\mathrm{U}$.

NOTE. Theorem 14 may be written in the form:

$$\frac{\mathrm{D}_x(\mathrm{U}\cdot\mathrm{V}\cdot\mathrm{W}\cdots)}{\mathrm{U}\cdot\mathrm{V}\cdot\mathrm{W}\cdots} = \frac{\mathrm{D}_x\mathrm{U}}{\mathrm{U}} + \frac{\mathrm{D}_x\mathrm{V}}{\mathrm{V}} + \frac{\mathrm{D}_x\mathrm{W}}{\mathrm{W}} + \cdots.$$

THEOR. 15. *The derivative, as to any variable, of a fraction whose terms are functions of that variable, is a fraction whose numerator is the product of the denominator into the derivative of the numerator less the product of the numerator into the derivative of the denominator, and whose denominator is the square of the given denominator.*

Let U, V be any functions of a variable x, and X their quotient; then will $\mathrm{D}_x\mathrm{X} = \dfrac{\mathrm{V}\cdot\mathrm{D}_x\mathrm{U} - \mathrm{U}\cdot\mathrm{D}_x\mathrm{V}}{\mathrm{V}^2}$.

For let x take any infinitesimal increment h, and x' be the new value of x, so that $x' \equiv x + h$, $\mathrm{X}' \equiv \mathrm{X} + \text{inc}\,\mathrm{X}$, $\mathrm{U}' \equiv \mathrm{U} + \text{inc}\,\mathrm{U}$, $\mathrm{V}' \equiv \mathrm{V} + \text{inc}\,\mathrm{V}$,

then $\because\ \mathrm{X} \qquad = \dfrac{\mathrm{U}}{\mathrm{V}}$ always,

$$\therefore\ \mathrm{X}' = \frac{\mathrm{U}'}{\mathrm{V}'},$$

i.e., $\qquad \mathrm{X} + \text{inc}\,\mathrm{X} = \dfrac{\mathrm{U} + \text{inc}\,\mathrm{U}}{\mathrm{V} + \text{inc}\,\mathrm{V}},$

$$\therefore \quad \text{inc}\,\text{X} = \frac{\text{U} + \text{inc}\,\text{U}}{\text{V} + \text{inc}\,\text{V}} - \frac{\text{U}}{\text{V}}$$

$$= \frac{\text{V} \cdot \text{inc}\,\text{U} - \text{U} \cdot \text{inc}\,\text{V}}{\text{V}^2 + \text{V} \cdot \text{inc}\,\text{V}},$$

$$\therefore \quad \frac{\text{inc}\,\text{X}}{h} = \frac{\text{V} \cdot \dfrac{\text{inc}\,\text{U}}{h} - \text{U} \cdot \dfrac{\text{inc}\,\text{V}}{h}}{\text{V}^2 + \text{V} \cdot \text{inc}\,\text{V}},$$

$$\therefore \quad \text{D}_x \frac{\text{U}}{\text{V}} = \frac{\text{V} \cdot \text{D}_x \text{U} - \text{U} \cdot \text{D}_x \text{V}}{\text{V}^2}. \qquad \text{Q.E.D.} \quad [\text{th.}7, \text{th.}8, \text{cr.}1$$

COR. 1. *If* U *be constant and* V *a function of* x,

$$\text{D}_x \text{U} = 0, \quad \textit{and} \quad \text{D}_x \frac{\text{U}}{\text{V}} = \frac{-\text{U} \cdot \text{D}_x \text{V}}{\text{V}^2} = \text{U} \cdot \text{D}_x \frac{1}{\text{V}}.$$

COR. 2. *If* V *be constant and* U *a function of* x,

$$\text{D}_x \text{V} = 0, \quad \textit{and} \quad \text{D}_x \frac{\text{U}}{\text{V}} = \frac{\text{D}_x \text{U}}{\text{V}} = \frac{1}{\text{V}} \cdot \text{D}_x \text{U}.$$

NOTE. Theorem 15 may be written in the form:

$$\text{D}_x \frac{\text{U}}{\text{V}} = \text{D}_x \left(\text{U} \cdot \frac{1}{\text{V}}\right) = \frac{1}{\text{V}} \cdot \text{D}_x \text{U} + \text{U} \cdot \text{D}_x \frac{1}{\text{V}}.$$

THEOR. 16. *The derivative, as to any variable, of a function of a function of that variable, is the product of the derivatives of the immediate functions which compose it, each taken as to the variable on which the immediate function depends.*

Let U be any immediate function of a variable x, and X any immediate function of U;

then will $\text{D}_x \text{X} = \text{D}_\text{U} \text{X} \cdot \text{D}_x \text{U}$.

For let x take an infinitesimal increment h, then U and X will take corresponding increments;

$$\text{and} \quad \because \quad \frac{\text{inc}\,\text{X}}{h} = \frac{\text{inc}\,\text{X}}{\text{inc}\,\text{U}} \cdot \frac{\text{inc}\,\text{U}}{h}, \qquad [\text{II. th.}3, \text{crs.}1, 7$$

$$\therefore \quad \lim \frac{\text{inc}\,\text{X}}{h} = \lim \frac{\text{inc}\,\text{X}}{\text{inc}\,\text{U}} \cdot \lim \frac{\text{inc}\,\text{U}}{h}, \text{ when } h \doteq 0, \quad [\text{th.}6, \text{cr.}, \text{th.}8$$

$$\textit{i.e.,} \quad \text{D}_x \text{X} = \text{D}_\text{U} \text{X} \cdot \text{D}_x \text{U}. \qquad \text{Q.E.D.}$$

THEOR. 17. *The derivative, as to any variable, of a commensurable power of that variable, is the product of the given exponent into a power of the variable whose exponent is a unit less than the given exponent.*

Let x be any variable, and n any commensurable number; then will $D_x x^n = n x^{n-1}$.

(*a*) n *a positive integer:*

For $\because\ x^n$ = the product $x \cdot x \cdot x \cdots$, n-times repeated,

$\therefore\ D_x x^n = x^{n-1} \cdot D_x x + x^{n-1} \cdot D_x x + \cdots$, n-times repeated [th.14

$= n x^{n-1} \cdot D_x x.$

But $\because\ D_x x = 1,$

$\therefore\ D_x x^n = n x^{n-1}.$

(*b*) n *a positive fraction,* $\frac{p}{q}$; p, q *both positive integers:*

$$\text{For let } X \equiv x^{\frac{p}{q}},$$

$$\text{then } \because\ X^q = x^p,$$

$$\therefore\ q X^{q-1} \cdot D_x X = p x^{p-1}, \qquad [(a)$$

$$\therefore\ D_x X = \frac{p}{q} \cdot \frac{x^{p-1}}{X^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{\frac{p}{q}})^{q-1}} = \frac{p}{q} \cdot \frac{(x^{\frac{1}{q}})^{q(p-1)}}{(x^{\frac{1}{q}})^{p(q-1)}}$$

$$= \frac{p}{q}(x^{\frac{1}{q}})^{q(p-1)-p(q-1)} = \frac{p}{q} x^{\frac{p-q}{q}} = \frac{p}{q} \cdot x^{\frac{p}{q}-1}.$$

[II. th. 3, crs. 10, 11

i.e., $D_x x^n = n x^{n-1}$. Q.E.D.

(*c*) n *any negative number,* $^-$m:

$$\text{For let } X \equiv x^{-m},$$

$$\text{then } D_x X = D_x \frac{1}{x^m} = \frac{-1 \cdot D_x x^m}{x^{2m}} \qquad [\text{th. 15, cr. 1}$$

$$= \frac{-m x^{m-1}}{x^{2m}} = -m x^{-m-1}, \qquad [(a), (b)$$

i.e., $D_x x^n = n x^{n-1}$. Q.E.D.

COR. *If* U *be any function of* x, *then*

$D_x U^n = n U^{n-1} \cdot D_x U.$ [th. 16

§ 10. INDETERMINATE FORMS.

If there be an expression that, by the definitions of the symbols used, may take an infinite number of different values lying in a continuous series, such an expression is *indeterminate*.

[See II. § 2, p. 28.

E.g., the expressions $0:0$, $\infty:\infty$, $\infty-\infty$, are indeterminate.

For the quotient $0:0$ may be any quotient that, multiplied by or into the divisor 0, gives the dividend 0 as product;

and any finite quotient may do this. [I. § 9 df. div.

And the quotient $\infty:\infty$ may be any quotient that, multiplied by or into the divisor ∞, gives the dividend ∞ as product;

and any quotient, not 0, may do this.

And the remainder $\infty-\infty$ may be any remainder that, added to the subtrahend ∞, gives the minuend ∞ as sum;

and any remainder may do this.

So the quotient $x:y$ is indeterminate if of x, y it be known only that both $\doteq 0$, or that both $\doteq \infty$.

And the remainder $x-y$ is indeterminate if of x, y it be known only that both $\doteq \infty$.

For any number may be such a quotient or remainder.

If for a particular value of any variable of which its terms are functions a fraction take the form $0:0$, it may be regarded as approaching this form by gradual change of the variable, and its *true value* is strictly the limit of the ratio of two infinitesimals.

This value is finite when the terms of the fraction are infinitesimals of the same order [§§ 4, 5], and it is indeterminate only so long as the law is unknown subject to which they $\doteq 0$.

E.g., when $x=1$, $x^3-1:x^2-1$ becomes $0:0$,

but when $x=1+h$, this fraction becomes

$$\frac{(1+h)^3-1}{(1+h)^2-1}, = \frac{h^3+3h^2+3h}{h^2+2h} = \frac{h^2+3h+3}{h+2},$$

$$\doteq 3:2 \text{ when } h \doteq 0, \quad i.e., \text{ when } x \doteq 1,$$

and its true value, when $x=1$, is $3:2$.

The reader will see that this process is equivalent to reducing the given fraction to its lowest terms, then substituting 1 for x.

In general, fractions take the form $0:0$ because of some common factor of their terms that vanishes for a particular value of the variable. If this factor can be found and divided out, and the particular value be substituted for the variable, the result is the true value of the given fraction; and this method is particularly useful for fractions whose terms are entire.

In the above example the vanishing factor is $x-1$, and the fraction, when this factor is divided out, becomes $\dfrac{x^2+x+1}{x+1}=\dfrac{3}{2}$.

Theorem 18 will show another method of evaluation.

Expressions that approach the forms $\infty:\infty$, $\infty-\infty$, may be reduced to equivalent expressions that approach the form $0:0$.

E.g., let x, x' be functions of any same variable x, such that when $x \doteq a$, then also x, x' both $\doteq \infty$.

Put x, x' under the forms $\mathrm{u}:\mathrm{v}$, $\mathrm{u}':\mathrm{v}'$,

wherein $\mathrm{u}, \mathrm{u}', \mathrm{v}, \mathrm{v}'$ are all functions of x such that, when $x \doteq a$, v is an infinitesimal of any order, and u is finite or an infinitesimal of a lower order than v, [th. 11, cr. 2

and v' is an infinitesimal of any order, and u' is finite or an infinitesimal of a lower order than v',

then $\mathrm{x}-\mathrm{x}', \doteq \infty-\infty, \equiv \dfrac{\mathrm{u}}{\mathrm{v}}-\dfrac{\mathrm{u}'}{\mathrm{v}'}=\dfrac{\mathrm{u}\cdot\mathrm{v}'-\mathrm{u}'\cdot\mathrm{v}}{\mathrm{v}\cdot\mathrm{v}'}, \doteq \dfrac{0}{0}.$

E.g., if $\mathrm{u}, \mathrm{v}, \mathrm{u}', \mathrm{v}' \equiv x+2,\ x-1,\ x^2-1,\ x^3-2x^2+x$,

then $\because$ u is finite, v, u' are infinitesimals of the first order, and v' is an infinitesimal of the second order, when $x \doteq 1$,

$$\begin{aligned}\therefore\ \mathrm{x}-\mathrm{x}', &\doteq \infty-\infty,\\ &=(\overline{x+2}\cdot\overline{x^3-2x^2+x}-\overline{x-1}\cdot\overline{x^2-1}):x(x-1)^3,\\ &\doteq 0:0,\\ &=(x^2-x-1):x-1, \doteq \infty, \text{ when } x \doteq 1.\end{aligned}$$

[div. out van. fac. $(x-1)^2$.

It has been shown above that the forms called *indeterminate* belong to that class of limiting expressions wherein the variables cease to have finite values. They differ from other limiting ex-

pressions of the same class in this, that their limits cannot be determined without more knowledge of the relations of the variables than appears upon the face of the expressions themselves.

E.g., when x, y both $\doteq 0$, the quotient $(8-x):(4-y)$, not an indeterminate form, $\doteq 2$, no matter how x, y may be related;

but the quotient $x:y$, $\doteq 0:0$, may have any limit whatever, depending on the relations of the variables x, y.

From this point of view the form $\infty \cdot 0$ may be added to the list of indeterminate forms; for although it does not, like the other three forms, take an infinite number of different values by the mere definition of the symbols taken absolutely, yet, like them, it may take any value whatever, considered as a limiting expression, *i.e.* as the limit of the product of any two variables, one of which $\doteq \infty$ and the other $\doteq 0$.

An expression that approaches the form $\infty \cdot 0$ may be reduced to an equivalent expression that approaches the form $0:0$.

THEOR. 18. *If for a particular value of a variable two functions of that variable both vanish, the true value of their quotient is the quotient of the values of their derivatives for that value of the variable.*

Let x, x' be two functions of a variable x such that $\mathrm{x}_a, \mathrm{x}'_a$, their values when a is put for x, both vanish;

then will $\mathrm{x}_a : \mathrm{x}'_a = \mathrm{D}_x\mathrm{x}_a : \mathrm{D}_x\mathrm{x}'_a$.

For, in x, x', put $a+h$ for x;

then $\because \mathrm{x}_{a+h} = \mathrm{x}_{a+h} - \mathrm{x}_a$, $\quad [\mathrm{x}_a = 0$

and $\mathrm{x}'_{a+h} = \mathrm{x}'_{a+h} - \mathrm{x}'_a$, $\quad [\mathrm{x}'_a = 0$

$$\therefore \mathrm{x}_{a+h} : \mathrm{x}'_{a+h} = \mathrm{x}_{a+h} - \mathrm{x}_a : \mathrm{x}'_{a+h} - \mathrm{x}'_a$$

$$= \frac{\mathrm{x}_{a+h} - \mathrm{x}_a}{h} : \frac{\mathrm{x}'_{a+h} - \mathrm{x}'_a}{h},$$

$$\therefore \lim(\mathrm{x}_{a+h} : \mathrm{x}'_{a+h})$$

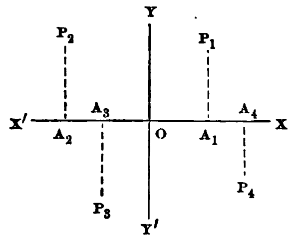
$$= \lim\left(\frac{\mathrm{x}_{a+h} - \mathrm{x}_a}{h} : \frac{\mathrm{x}'_{a+h} - \mathrm{x}'_a}{h}\right), \text{when } h \doteq 0,$$

$$= \lim \frac{\mathrm{x}_{a+h} - \mathrm{x}_a}{h} : \lim \frac{\mathrm{x}'_{a+h} - \mathrm{x}'_a}{h}, \quad [\text{th. 8, cr. 1}$$

i.e., $\mathrm{x}_a : \mathrm{x}'_a = \mathrm{D}_x\mathrm{x}_a : \mathrm{D}_x\mathrm{x}'_a$. Q.E.D.

§ 11. GRAPHICAL REPRESENTATION OF FUNCTIONS.

For convenience in treating of integration and other subjects discussed later, the geometric words, origin, axis, abscissa, and ordinate, are here defined, and the reader is introduced to the method of representing by a geometric locus an algebraic equation between two variables, or a function of a single variable.



Let P be any point, X'X any straight line lying in the plane of the paper, and O a fixed point on X'X; from P draw PA perpendicular to X'X and meeting it at A; then X'X is the *reference line*, or *axis*, the fixed point O is the *reference point*, or *origin*, the line OA is the *abscissa* of the point P, AP is its *ordinate*, and OA, AP together are the *coördinates* of P.

If the figure lie before the reader so that X'X is a horizontal line with X to the right of X', then the direction X'X is ordinarily taken as the positive direction and XX' as the negative direction [I. § 3]; and abscissas measured to the right from O are positive, while those measured to the left are negative. So, ordinates measured upward from the axis are positive, and those measured downward are negative.

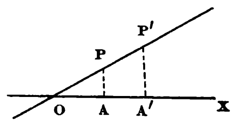
An abscissa is generally represented by the letter x, and an ordinate by y. So, the line X'X is called the *axis of abscissas*, or the *axis of* x; and the line Y'Y, drawn perpendicular to X'X through O, is called the *axis of ordinates*, or the *axis of* y.

The position of a point is determined, and the point may be constructed, when its coördinates are given.

When the coördinates are not given, but are connected by a given relation (*e.g.*, that their sum is constant), an infinite number of points may be found that satisfy the conditions, for if any value be assumed for the abscissa, the given relation between the coördinates serves to determine the corresponding value or values of the ordinate.

In general, these points all lie in some line, straight or curved, called their *locus;* and the relation between the variable coördinates may be expressed by a single equation between two variables, called the *equation of the locus.* In this equation either variable is a function of the other. The equation is satisfied by the coördinates of every point of the locus, and by those of no other point. Such equations are generally written in the form $y = fx$, wherein x, the abscissa, may be regarded as an independent variable, and y, the ordinate, as a function of x; and the shape of the locus of the extremities of the ordinates shows the manner in which fx varies with x.

E.g., the locus of points whose coördinates satisfy the relation expressed by the equation $y = mx$ is a straight line through the origin.



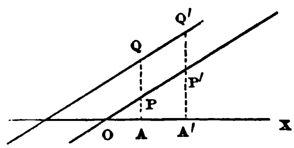
Let OX be the axis, O the origin, P, P' any two points whose coördinates OA, AP, OA', A'P' are so related that $AP = m \cdot OA$, and $A'P' = m . OA'$, *i.e.*, so that $y = mx$ for each of them; then is PP' a straight line through O.

For ∵ AP : OA = A'P' : OA', [hyp.

and ∵ AP is parallel to A'P', [constr.

∴ the straight line OP passes through P', and is the locus sought. [geom.

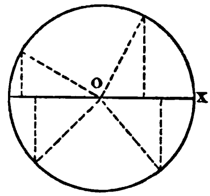
So, the locus of the equation $y = mx + b$ is a straight line that cuts the axis of y at a distance b above the origin.



As above, construct the straight line that represents the equation $y = mx$; draw any two ordinates AP, A'P', and extend them to Q, Q', so that $AQ = AP + b$, $A'Q' = A'P' + b$, wherein b is any constant;

then is QQ' a straight line parallel to OPP', and the locus sought.

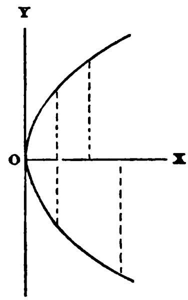
So, the locus of the equation $x^2+y^2=r^2$, wherein r is constant, is a circle whose centre is the origin and whose radius is r.

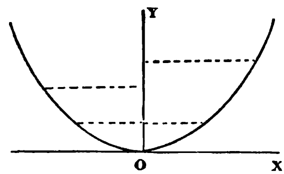


The reader may see this from the principle of geometry that "in a right triangle the square of the hypotenuse equals the sum of the squares of the other two sides."

So, the locus of the equation $y^2=px$ is a parabola whose axis is the axis of x, whose vertex is at the origin, and whose parameter is p;

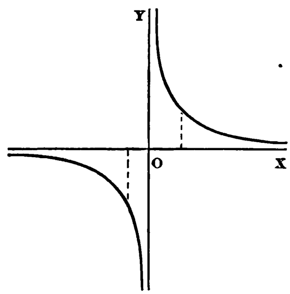
and the locus of the equation $x^2=py$ is a parabola whose axis is the axis of y.





The reader will recognize these equations as the algebraic expression of the geometric property of the parabola, that "the square of a perpendicular from any point of the curve to its axis equals the product of its parameter into that part of the axis intercepted between the vertex and the foot of the perpendicular."

So, the locus of the equation $xy=c^2$ is the rectangular hyperbola, taken with reference to its asymptotes as axes of coördinates.



These figures also represent graphically the functions mx, $mx+b$, $\sqrt{(r^2-x^2)}$, $\sqrt{px}$. $\frac{x^2}{p}$, $\frac{c^2}{x}$, and show how they vary with x.

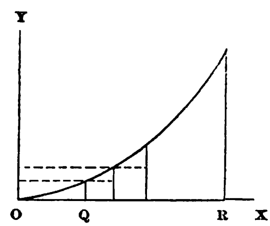
§ 12. INTEGRATION.

THEOR. 19. *If there be a variable* x, *and if* fx *be a function of* x *whose derivative as to* x *is* f′x *and is continuous; and if the variable begin with the value* x_0, = a, *and take* n *more successive values* $x_1, x_2, \cdots x_n$, = b; *and if, while* a *and* b *stand fast,* $n \doteq \infty$ *and* $x_1 - x_0,\ x_2 - x_1, \cdots$ *each* $\doteq 0$; *then the sum of the series of products* $(x_1-x_0)f'x_0,\ (x_2-x_1)f'x_1, \cdots (x_n-x_{n-1})f'x_{n-1}$, *approaches* fb − fa *as its limit.*

ILLUSTRATIVE EXAMPLES.

That the reader may clearly understand the meaning of the theorem and its proof, and that he may see how this method of summation was first suggested, and follow the historical order of investigation, special applications of it to the finding of areas and volumes are given before the formal proof:

To find the area of the figure included between two given ordinates, the axis of abscissas, and the parabolic curve whose equation is $x^2 = py$:



Let the two given ordinates correspond to the abscissas OQ, $= a$, and OR, $= b$; divide QR into n parts; let the abscissas of the $n+1$ successive points (including Q and R) be $x_0, x_1, x_2, \cdots x_n$; and the corresponding ordinates $y_0, y_1, y_2, \cdots y_n$, and let n rectangles be formed as in the figure;

then ∵ s, the area sought, is the limit of the sum of $s_1, s_2, s_3, \cdots s_n$, the areas of the n rectangles, when $n \doteq \infty$ and $x_1 - x_0,\ x_2 - x_1, \cdots$ each $\doteq 0$, [th. 12, nt.

and ∵ $s_1 = y_0 \cdot (x_1 - x_0)$ [geom.

$= \frac{1}{p} \cdot x_0^2 \cdot (x_1 - x_0)$, [hyp.

and ∵ $3x_0^2, = D_{x_0} x_0^3$,

$= \lim \frac{x_1^3 - x_0^3}{x_1 - x_0}$ when $x_1 - x_0 \doteq 0$, [§ 5, df. deriv.

$$\therefore 3x_0^2 = \frac{x_1^3 - x_0^3}{x_1 - x_0} + \epsilon_1,$$

wherein ϵ_1 is some variable that $\doteq 0$ when $x_1 - x_0 \doteq 0$,

and $\quad s_1 = \frac{1}{3p}[x_1^3 - x_0^3 + (x_1 - x_0)\epsilon_1]$.

So $\quad s_2 = \frac{1}{3p}[x_2^3 - x_1^3 + (x_2 - x_1)\epsilon_2]$,

$\quad s_3 = \frac{1}{3p}[x_3^3 - x_2^3 + (x_3 - x_2)\epsilon_3]$,

$\quad \cdots \qquad \cdots$

and $\quad s_n = \frac{1}{3p}[x_n^3 - x_{n-1}^3 + (x_n - x_{n-1})\epsilon_n]$,

wherein $\epsilon_1, \epsilon_2, \cdots$ each $\doteq 0$ when $n \doteq \infty$ and $x_1 - x_0, \cdots \doteq 0$;

$$\therefore \Sigma s = \frac{1}{3p}[x_n^3 - x_0^3 + \Sigma(x_1 - x_0)\epsilon_1].$$

But $\because \Sigma(x_1 - x_0)\epsilon_1 \not> (x_n - x_0)\epsilon_r$,

wherein ϵ_r is the largest of the ϵ's,

$\therefore \Sigma(x_1 - x_0)\epsilon_1 \doteq 0$ when $\Sigma s \doteq \text{s}$,

$$\therefore \text{s} = \frac{1}{3p}(x_n^3 - x_0^3)$$

$$= \frac{1}{3p}(b^3 - a^3). \qquad \text{Q.E.F.}$$

So to find the volume of the solid generated by the same figure revolving about ox, the tangent at the vertex:

then $\because$ v, the volume sought, is the limit of the sum of $v_1, v_2, v_3, \cdots v_n$, the volumes of the n cylinders of revolution generated by the n rectangles when $n \doteq \infty$ and $x_1 - x_0, x_2 - x_1, \cdots$ each $\doteq 0$, [th. 12

and $\because v_1 = \pi y_0^2 (x_1 - x_0)$ [geom.

$= \frac{\pi}{5p^2} \cdot 5x_0^4(x_1 - x_0)$ [hyp.

$= \frac{\pi}{5p^2}\left(\frac{x_1^5 - x_0^5}{x_1 - x_0} + \epsilon_1\right)(x_1 - x_0)$ [§ 5, df. deriv.

$= \frac{\pi}{5p^2}[x_1^5 - x_0^5 + (x_1 - x_0)\epsilon_1]$,

and $\because\ v_2 = \frac{\pi}{5p^2}[(x_2^5 - x_1^5) + (x_2 - x_1)\epsilon_2]$,

$v_3 = \frac{\pi}{5p^2}[(x_3^5 - x_2^5) + (x_3 - x_2)\epsilon_3]$,

$\cdots \qquad \cdots$

$v_n = \frac{\pi}{5p^2}[(x_n^5 - x_{n-1}^5) + (x_n - x_{n-1})\epsilon_n]$,

$\therefore\ \Sigma v = \frac{\pi}{5p^2}[(x_n^5 - x_0^5) + \Sigma(x_1 - x_0)\epsilon_1]$;

and $\mathrm{V} = \frac{\pi}{5p^2}(b^5 - a^5)$. Q.E.F. [as above

PROOF OF THE THEOREM.

For $\because\ \frac{fx_1 - fx_0}{x_1 - x_0} \doteq f'x_0$, when $x_1 - x_0 \doteq 0$, [§ 5, df. deriv.

$\therefore\ \frac{fx_1 - fx_0}{x_1 - x_0} = f'x_0 + \epsilon_1$,

wherein ϵ_1 is some variable that $\doteq 0$ when $x_1 - x_0 \doteq 0$,

$\therefore\ fx_1 - fx_0 = (x_1 - x_0)\cdot(f'x_0 + \epsilon_1)$.

So $fx_2 - fx_1 = (x_2 - x_1)\cdot(f'x_1 + \epsilon_2)$,

$fx_3 - fx_2 = (x_3 - x_2)\cdot(f'x_2 + \epsilon_3)$,

$\cdots \qquad \cdots,$

and $fx_n - fx_{n-1} = (x_n - x_{n-1})(f'x_{n-1} + \epsilon_n)$;

wherein $\epsilon_1, \epsilon_2, \cdots$ each $\doteq 0$, when $n \doteq \infty$ and $x_1 - x_0, \cdots \doteq 0$,

$\therefore\ fx_n - fx_0 = (x_1 - x_0)f'x_0 + \cdots + (x_n - x_{n-1})f'x_{n-1}$
$+ (x_1 - x_0)\epsilon_1 + \cdots + (x_n - x_{n-1})\epsilon_n$. [II. ax. 2

But $\because\ \Sigma(x_1 - x_0)\epsilon_1 \not> (x_n - x_0)\epsilon_r$,

wherein ϵ_r is the largest of the ϵ's,

and $\because\ \epsilon_r \doteq 0$, when $n \doteq \infty$ and $x_1 - x_0, \cdots$ each $\doteq 0$, [above

$\therefore\ fb - fa = \lim[(x_1 - x_0)f'x_0 + \cdots + (x_n - x_{n-1})f'x_{n-1}]$. Q.E.D.

NOTE. The theorem may be written in the form:

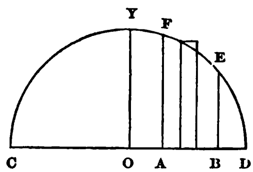
$\lim \Sigma_a^b f'x \cdot \mathrm{inc}\, x = fb - fa$, when $\mathrm{inc}\, x \doteq 0$,

wherein $\Sigma_a^b f'x \cdot \mathrm{inc}\, x \equiv f'x_0 \cdot \mathrm{inc}\, x_0 + \cdots + f'x_{n-1} \cdot \mathrm{inc}\, x_{n-1}$,

and $x_0 \equiv a, \quad x_n \equiv b, \quad \mathrm{inc}\, x_r \equiv x_{r+1} - x_r$.

EXAMPLES OF THE DIRECT APPLICATION OF THE THEOREM.

To find V, the volume of a segment of a sphere of radius r, whose bounding planes are distant a, b from the centre.



Let CDY be a semicircle of radius r; take CD for the axis of x; let AF, BE be two ordinates distant a, b from the centre O; and let the whole revolve about CD;

then the area of ABEF is the limit of the sum of the areas of a large number of rectangles;

and V, the volume of the solid generated by ABEF, is the limit of the sum of the volumes of the corresponding cylinders of revolution.

Take $\pi y_p^2(x_{p+1}-x_p), =\pi(r^2-x_p^2)(x_{p+1}-x_p)$, as the type-term of this series;

then $\because f'x=\pi(r^2-x^2)$,

$\therefore fx =\pi(r^2x-\frac{1}{3}x^3)$,

i.e., $V =\pi[(r^2b-\frac{1}{3}b^3)-(r^2a-\frac{1}{3}a^3)]$ $[V=fb-fa$

$=\pi(b-a)[r^2-\frac{1}{3}(a^2+ab+b^2)]$

$=\frac{1}{6}\pi(b-a)[(r^2-a^2)+(r^2-b^2)+(4r^2-\overline{a+b}^2)]$

$=\frac{1}{6}\pi c(r_1^2+r_2^2+4r_3^2)$,

wherein c is the thickness of the segment, r_1, r_2, r_3 the radii of its bases and middle section. Q.E.F.

So the volume of the hemisphere generated by the quadrant ODY

$=\pi(fr-f0)=\pi(r^3-\frac{1}{3}r^3)=\frac{2}{3}\pi r^3$.

And the whole volume of the sphere

$=\frac{4}{3}\pi r^3$. Q.E.F.

NOTE. If in $\frac{1}{6}\pi c(r_1^2+r_2^2+4r_3^2)$, the general expression for the volume of a spherical segment, r be put for r_1, 0 for r_2, $\frac{1}{2}r\sqrt{3}$ for r_3, r for c, the result is the volume of the hemisphere; and if 0 be put for r_1, 0 for r_2, r for r_3, $2r$ for c, the result is the volume of the sphere. The results thus found are identical with those given above.

So to find H, the height fallen through in a given time by a body starting from rest, on the assumption that, within any distance required in practice, the velocity of a falling body increases uniformly, and hence that the velocity acquired at any instant is proportional to the time of falling from rest; [laws of motion

then $v = gt$, wherein g is some constant; v is the velocity at the end of t seconds from starting, *i.e.*, the number of feet the body would fall through in the next second if its rate did not change during that second.

Let the entire time, T, be divided into n intervals, ending respectively at $t_1, t_2, \cdots t_n$ seconds from starting, and let $t_0 = 0$, and $t_n = \mathrm{T}$; assume that the velocity during each interval of time remains constant at what, under the laws of motion, it should be at the beginning of the interval, and let $h_1, h_2, \cdots h_n$ stand for the distances fallen through in the 1st, 2d, $\cdots$ nth intervals,

then $\because$ $h_1 = v_0(t_1 - t_0) \quad = g t_0(t_1 - t_0)$,

$h_2 = v_1(t_2 - t_1) \quad = g t_1(t_2 - t_1)$,

$\cdots \qquad \cdots$

$h_r = v_r(t_r - t_{r-1}) = g t_r(t_r - t_{r-1})$,

$\cdots \qquad \cdots$,

$\therefore$ $\mathrm{H} = g \cdot \lim \Sigma_0^{\mathrm{T}} t \cdot \mathrm{inc}\, t$ when $\mathrm{inc}\, t \doteq 0$ [th. 19, nt.

$= g \cdot \frac{1}{2}(\mathrm{T}^2 - 0^2)$ [$f't = t,\ ft = \frac{1}{2}t^2$

$= \frac{1}{2} g \mathrm{T}^2$. Q.E.F.

So to find the *ultimate average* (*i.e.* the limit of the quotient of the sum of a series of terms by their number when that number becomes infinite) of the successive values taken by fx as x increases from the value a to the value b:

Let x take n successive values a, $a + h$, $a + 2h$, $\cdots b$, *i.e.*, let x increase by $n - 1$ equal increments h from a to b, and let fx take the corresponding values fa, $f(a + h)$, $f(a + 2h)$, $\cdots fb$,

then $\because$ the average of these values is $(fa + \cdots + fb) : n$,

i.e., $(h \cdot fa + \cdots + h \cdot fb) : (b - a + h)$, [$n - 1 = (b - a) : h$

$\therefore$ their ultimate average, when $h \doteq 0$, is $\lim \Sigma_a^b h \cdot fx : (b - a)$. [th. 19

§ 13. EXAMPLES.

§ 9. PROB. 1.

··· 2. Apply the increment h to x in the following functions; find the corresponding increments of the functions, and thence their derivatives:

1. x^3; x^4; x^{-2}; x^{-3}; $ax+b$; ax^2-bx^3; $ax+bx^{-1}$; $ax^{-2}+bx^4$.

2. $\dfrac{x}{a}$; $\dfrac{c}{x}$; $\dfrac{d}{x^2}$; $\dfrac{ex^2+f}{x^3}$; $\dfrac{ax^{-1}+b}{x}$; $\dfrac{ax^2+2bx+c}{x^2}$; $\dfrac{x^2-3x+2}{(x-1)^2}$.

THEOR. 13.

··· 5. Expand, where necessary, and find the derivatives of:

3. $2x^2+3x+5$; $(x+1)(x+2)$; $x^3+3ax^2+3a^2x+a^3$.

4. $x+x^{-1}$; $(x+x^{-1}+1)^2$; $(1-x+x^2)^2-(1+x-x^2)^2$.

5. $\left(\dfrac{1}{a}+\dfrac{x}{b}\right)^2$; $\dfrac{a}{x}+\dfrac{b}{x^2}+\dfrac{c}{x^3}$; $\left(1-\dfrac{1}{x}-\dfrac{1}{x^2}\right)^2-\left(1+\dfrac{1}{x}-\dfrac{1}{x^2}\right)^2$.

6. Of what are the following expressions derivatives as to x?
$3x^2$; $x+2x^2+3x^3$; $ax+b$; x^{-2}; ax^2-bx^3; $ax^{-2}-bx^{-3}$.

THEOR. 14.

··· 9. Find the derivatives of:

7. $x(x+1)$; $-3x(ax+b)$; $(2x+1)(3x+2)$; $(3x+1)^2$.

8. $x(x+1)(x+2)$; $(x+1)^2(x+2)$; $x^2(x+1)$; $x^2(x^2-9)$.

9. $x(a-2x)(2a+3x)$; $(x+a)^2(x+b)^2$; $(x-a)^4$.

10. Of what are the following expressions derivatives as to x?
$2x(x+1)+x^2$; $3(2x+1)+2(3x+2)$; $3(x^2+3)+6x^2$;
$(x+a)(x+b)+(x+b)(x+c)+(x+c)(x+a)$.

THEOR. 15.

··· 12. Find the derivatives of:

11. $\dfrac{1}{x}$; $\dfrac{a}{x^2}$; $\dfrac{x+1}{x}$; $\dfrac{x}{x+1}$; $\dfrac{x-1}{x+1}$; $\dfrac{x+a}{x+b}$; $\dfrac{a-x}{a+x}$; $\dfrac{x^2-ax+b}{x-a}$.

12. $\dfrac{3x-1}{x^3}$; $\dfrac{x^2-px+q}{x^2+px+q}$; $\dfrac{x-2}{x^2-4x+6}$; $\dfrac{(x+2)^2-(x-1)^2}{(x-2)^2-(x+1)^2}$.

13. Of what are the following expressions derivatives as to x?
$\dfrac{-1}{x^2}$; $\dfrac{x-(x+1)}{x^2}$; $\dfrac{b(c+dx)-d(a+bx)}{(c+dx)^2}$; $\dfrac{bc-ad}{(c+dx)^2}$.

THEORS. 16, 17.

... 17. Find the derivatives of:

14. $(a+bx^3)^4$; $(1+x^n)^{-m}$; $(a^{\frac{1}{2}}+2x^{\frac{1}{2}})^{-2}-(a^{\frac{1}{2}}-2x^{\frac{1}{2}})^{-2}$.

15. $(1+x^2)^{\frac{1}{2}}$; $(a^m-x^{-m})^{-\frac{1}{m}}$; $\sqrt{(ax^2+2bx+c)}$; $f(x^2+px+q)$.

16. $\sqrt{(x-1)}-\dfrac{1}{\sqrt{(x-1)}}$; $\sqrt{\dfrac{x^2-1}{(x^2+1)^3}}$; $\dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$.

17. $\dfrac{x}{\sqrt{(1-x^2)}}\cdot\left(\dfrac{x}{1+\sqrt{(1-x^2)}}\right)^n$; $\left(\dfrac{x^2-3x+5}{x^2-3x-5}\right)^{\frac{1}{2}}$; $\dfrac{1}{x^2}\cdot f\dfrac{x+a}{b-x}$.

... 20. By finding the derivative, D^2, of the first derivative D^1, then the derivative, D^3, of D^2, and so on, show that:

18. $D_x^3(x^3)=6$; $D_x^4(x^4)=24$; $D_x^n(x^n)=n\,!$; $D_x^4(x^6)=360x^2$.

19. $D_x^m(x^n)=n(n-1)(n-2)\cdots(n-m+1)x^{n-m}$. Find $D_x^3(x^{-n})$.

20. $D_x^2\sqrt{(x^2+a^2)}=\dfrac{a^2}{(x^2+a^2)^{\frac{3}{2}}}$; $D_x^n\dfrac{1-x}{1+x}=\dfrac{2(-1)^n\cdot n\,!}{(1+x)^{n+1}}$.

21. Of what are the following expressions derivatives as to x?
$4(a+bx^3)^3\cdot 3bx^2$; $12bx^2(a+bx^3)^3$; $-10x(1-x^2)^4$.

22. Show that $X^n\cdot D_xX$ is the derivative of $\dfrac{X^{n+1}}{n+1}$; thence find the expressions whose derivatives, as to x, are:
$2x(a^2+x^2)^n$; $x\sqrt{(a^2+x^2)}$; $(ax+b)\sqrt{(ax^2+2bx+c)}$;
$\dfrac{D_xX}{X^n}$; $\dfrac{x^2-2x+2}{(x^3-3x^2+6x-1)^5}$; $\dfrac{x}{(a^2+x^2)^{\frac{1}{2}}}$; $\dfrac{x^{\frac{9}{2}}}{(x^3+x^7)^{\frac{1}{2}}}$;
$\dfrac{ax+b}{(ax^2+2bx+c)^{\frac{3}{2}}}$; $\dfrac{ax^2+2bx+c}{\sqrt{(ax^3+3bx^2+3cx+d)}}$.

§ 10. THEOR. 18.

23. By means of h. c. msrs. find the true value, when $x=2$, of:
$(x^2-5x+6):(x^2-6x+8)$; $(x^2-3x+2):(x^2-4)$.

24. So, when $x=1$, of:
$(x^3-x^2-3x+3):(x^3+x^2-x-1)$; $(x-1):(x^n-1)$.

25. So, when $x=\frac{1}{2}$, of:
$(4x^2-1):(32x^5-1)$; $(6x^4+x^3-x):(4x^3-6x^2-4x+3)$.

26. So, when $x = c$, of:
$(ax^2 - 2acx + ac^2) : (bx^2 - 2bcx + bc^2)$; $(x^3 - c^3)^{\frac{1}{2}} : (x^{\frac{1}{2}} - c^{\frac{1}{2}})$.

27. By means of derivatives find the true value, when $x = 1$, of:
$(x^3 + x^2 - 2) : (x^3 + 2x^2 - 2x - 1)$;
$(x^4 - 3x^3 + 2x^2 + x - 1) : (x^3 - x^2 - 2x + 2)$.

28. So, when $x = -1$, of:
$(x^3 + 1) : (x^3 + x^2 + x + 1)$; $(x^5 + 1) : (x^3 + 4x + 5)$.

29. So, when $x = \frac{7}{3}$, of:
$(3x^3 - 13x^2 + 23x - 21) : (6x^3 + x^2 - 44x + 21)$.

30. So, when $x = 2a$, of:
$(x^4 - ax^3 - a^2x^2 - a^3x - 2a^4) : (3x^3 - 7ax^2 + 3a^2x - 2a^3)$.

31. So, when $x = 0$, of:
$[1 - \sqrt{(1-x)}] : [\sqrt{(1+x)} - \sqrt{(1+x^2)}]$.

32. So, when $x = 1$, of:
$[(3x^3 - 2x^6)^{\frac{1}{2}} - x^{\frac{6}{5}}] : [1 - x^{\frac{2}{3}}]$; $[x^{\frac{3}{2}} - 1 + (x-1)^{\frac{3}{2}}] : [x^2 - 1]^{\frac{1}{2}}$.

33. So, when $x = a$, of:
$[(a^2 - x^2)^{\frac{1}{2}} + (a - x)^{\frac{2}{3}}] : [(a^3 - x^3)^{\frac{1}{2}} + (a - x)^{\frac{1}{2}}]$;
$[\sqrt{(a + x)} - \sqrt{(2x)}] : [\sqrt{(a + 3x)} - 2\sqrt{x}]$;
$[x^{\frac{1}{2}} - a^{\frac{1}{2}} + (x - a)^{\frac{1}{2}}] : [x^2 - a^2]^{\frac{1}{2}}$; $[fx - fa] : [\phi x - \phi a]$;
$[(c^2 - x^2)^{\frac{1}{2}} - (c^2 - a^2)^{\frac{1}{2}}] : [(c^2 - x^2)^{\frac{1}{3}} - (c^2 - a^2)^{\frac{1}{3}}]$.

34. Put $a + h$ for x, $a + k$ for y, expand and reduce, then let h, k become infinitesimal, and thereby find the true value, when $x = y = a$, of:
$[(x-y)a^n + (y-a)x^n + (a-x)y^n] : [(x-y)(y-a)(a-x)]$.

§ 11.

35. Draw the lines whose equations are:
$x = 3$; $y = 5$; $x = 0$; $y = x$; $y = 2x$; $y = 2x + 3$.

36. Plat the equations:
$(x + 3)^2 + y^2 = 16$; $x^2 + 9y^2 = 9$; $4x^2 - y^2 = 16$.

37. Trace the curve whose equation is $xy = 16$; and show that it has four infinite branches that continually approach the axes.

38. In the equation $y = x^3 - 5x^2 + 2x + 6$ give x the values -3, -2, -1, 0, 1, 2, 3, 4, 5, in succession; thence find the corresponding values of y, and plat the equation.

39. Represent graphically the functions:

$3x+5$; x^2+1; x^2+3x+2; $x^2+3x+2\frac{1}{4}$; x^2+3x+3.

40. Plat the functions: $5 \pm [9-(x+2)^2]^{\frac{1}{2}}$; $5 \pm [9+(x+2)^2]^{\frac{1}{2}}$.

41. If $P_0, (x_0, y_0)$, $P_1, (x_1, y_1)$ be points on the curve $y = f(x)$, show by the properties of similar triangles that the equation of the chord P_0P_1 is

$(y - y_0) : (x - x_0) = (y_1 - y_0) : (x_1 - x_0)$.

Let P_1 approach P_0 so that $x_1 \doteq x_0$, $y_1 \doteq y_0$, and show that the equation of the line tangent to the curve at P_0 is $y - y_0 = f'(x_0) \cdot (x - x_0)$; *e.g.*, that the equation of the tangent, at the point whose abscissa is 2, to the curve $y = 2x^3 - 3x^2 + 3x - 7$ is $y - 3 = 15(x - 2)$.

Plat this curve and the tangent.

42. Show that when $f'x_0 = 0$ the tangent at x_0, y_0 is parallel to the x-axis, *i.e.*, that the point x_0, y_0 is an *elbow* of the curve; and that when the plat of $f'x$ crosses the x-axis the plat of fx has an elbow.

43. Plat the function $x^4 - 4x^3 + x^2 + 7x - 3$ and its three derived functions.

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44. Find the area of the figure bounded by the axis of abscissas, the curve $y = x^2 + x + 1$, and the ordinates corresponding to the abscissas 2 and 3; find also the volume of the solid generated by the revolution of this figure about the axis of x.

45. So for $y = (x+1)(x+2)$, between the abscissas 1, 3.

46. So for $y = x^4 + 4x^3 + 2x^2 + 3$, between the abscissas 1, 2.

47. So for $x^4 + ax^3 + a^2x^2 + b^3y = 0$, between the abscissas a, 0.

48. Find the area enclosed by the axis of y, the lines $y = 1$, $y = 0$, and the curve $x^2(y^2 + 5y + 4) = (2y + 5)^2$.

49. Find the area of the figure enclosed by the two axes and the curve $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$; and find the volume of the solid generated by its revolution about either axis.

50. Find the area of the figure enclosed by the curve $xy^2=y^{\frac{5}{2}}+2y^{\frac{3}{2}}+5$ and the lines $x=0$, $y=0$, $y=1$.

51. Find the area of the figure cut off from the curve $y=(x+1)(x+2)$ by the axis of x.

52. If the figure enclosed by the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ and the axes revolve about either axis, find the volume of the solid generated.

53. Find the area of the figure cut from the curve $ay^2=x^3$ by the line $x=a$; and find the volumes of the solids generated by its revolution about that line; about the axis of x; and about the axis of y.

54. Find the volume of the ring generated by the circle $x^2+y^2=25$ revolving about the line $x=7$.

55. The curve $xy=c^2$ revolves about the axis of y. Show that the volume generated by the infinite branch beginning at the vertex (c, c) is equal to the volume of the cylinder generated by the ordinate of the vertex.

56. Find the ultimate average value of the function $3x^2+5x-7$ as x varies continuously from 1 to 4.

57. So for the function x^3-3x^2+2x-1 between $x=0$ and 3.

58. If the function named in Ex. 57 be platted, show that the result of that example gives the ultimate average length of equidistant ordinates between $x=0$ and $x=3$.

59. Find the average ordinate lying between the given ordinates in Examples 44–47; and show that for any figure with a rectilinear base the product of the average ordinate by the base is the area.

VIII. POWERS AND ROOTS.

§ 1. FRACTIONAL POWERS.

THE words *power*, *root*, *base*, *exponent*, and *root-index* are defined in I. § 10. A root-index is always assumed to be a positive integer; but an exponent may be any number whatever.

The value of a fractional power is commonly ambiguous.

E.g., $100^{\frac{1}{2}} = \pm 10$; $9^{-\frac{3}{2}} = \pm \frac{1}{27}$.

So, as appears later, every base except 0 has three distinct cube roots, four distinct fourth roots, and so on. Some of these roots, however, are neither purely positive nor purely negative; they are called *imaginaries*, or, better, *complexes*, and discussed in chap. X.

Different powers of a base are *in the same series* when they are integral powers of the same root. An integral power of a base belongs to all series alike.

E.g., 9^{-1}, $9^{-\frac{1}{2}}$, 9^{0}, $9^{\frac{1}{2}}$, 9^{1}, $9^{\frac{3}{2}}$, 9^{2}, $\cdots$

are the -2d, -1st, 0th, 1st, 2d, 3d, 4th, $\cdots$ powers of $\sqrt{9}$,

i.e., of $^{-}3$ and of $^{+}3$: they form the two series

$\frac{1}{9}$, $^{-}\frac{1}{3}$, 1, $^{-}3$, 9, $^{-}27$, 81, $\cdots$, powers of $^{-}\sqrt{9}$,

and $\frac{1}{9}$, $\frac{1}{3}$, 1, 3, 9, 27, 81, $\cdots$, powers of $^{+}\sqrt{9}$;

but the integral powers $\frac{1}{9}$, 1, 9, 81, belong to both series.

When several powers of the same base occur together, they are assumed to be all taken in the same series.

E.g., the value of $4^{-\frac{1}{2}} - 3 \cdot 4^{\frac{1}{2}} + 4^{\frac{3}{2}}$ is either

$$\frac{1}{^{+}2} - 3 \cdot {}^{+}2 + ({}^{+}2)^3, = 2\tfrac{1}{2}, \text{ or } \frac{1}{^{-}2} - 3 \cdot {}^{-}2 + ({}^{-}2)^3, = -2\tfrac{1}{2},$$

according as $4^{-\frac{1}{2}}$, $4^{\frac{1}{2}}$, $4^{\frac{3}{2}}$ are taken as powers of $^{+}2$ or of $^{-}2$;

but not $\frac{1}{^{+}2} - 3 \cdot {}^{-}2 + ({}^{+}2)^3, = 14\frac{1}{2}$, nor $\frac{1}{^{-}2} - 3 \cdot {}^{+}2 + ({}^{+}2)^3, = 1\frac{1}{2}$.

So, $\sqrt{9a^2} - \sqrt{4a^2} = \pm 3a \mp 2a = \pm a$,

but not $3a + 2a$, nor $-3a - 2a$.

Powers of different bases are *like powers* if they have the same exponent.

E.g., $\sqrt{a}$, $\sqrt{b}$, $\sqrt{ab}$; a^2, b^2, $\overline{ab}^2$; 2^n, 3^n, 6^n.

§ 2. COMBINATIONS OF COMMENSURABLE POWERS.

☞ THAT every commensurable power of a real positive base has at least one real value is shown from independent considerations in th. 5, which may therefore be read here if preferred.

THEOR. 1. *Any commensurable power of a base has the same value or values whether the exponent be in its lowest terms or not.*

Let k, p', q be any positive integers; p, $= \pm p'$, any integer, positive or negative; A, any base;

then will every value of $\mathrm{A}^{\frac{kp}{kq}}$ be a value of $\mathrm{A}^{\frac{p}{q}}$, and conversely.

For $\because$ $\sqrt[kq]{\mathrm{A}} \equiv$ one of any kq equal factors into which A can be resolved, [I. § 10 df. root

and $\because$ the product of any k of these factors is equal to that of any other k of them,

$\therefore$ all the q partial products so formed are equal, and each is a value of $\sqrt[q]{\mathrm{A}}$,

and every single value of $\sqrt[kq]{\mathrm{A}}$ is a value of $\sqrt[k]{(\sqrt[q]{\mathrm{A}})}$.

So $\because$ $[\sqrt[k]{(\sqrt[q]{\mathrm{A}})}]^{kq} = [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}] \cdot [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}] \cdots kq$ factors

$= [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}]^{k} \cdot [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}]^{k} \cdots$ q factors

$= (\sqrt[q]{\mathrm{A}}) \cdot (\sqrt[q]{\mathrm{A}}) \cdot (\sqrt[q]{\mathrm{A}}) \cdots$ q factors

$= \mathrm{A}$, [df. root

$\therefore$ every value of $\sqrt[k]{(\sqrt[q]{\mathrm{A}})}$ is a value of $\sqrt[kq]{\mathrm{A}}$.

$\therefore$ $\sqrt[kq]{\mathrm{A}} = \sqrt[k]{(\sqrt[q]{\mathrm{A}})}$,

i.e., every value of either member is a value of the other.

So $\because$ $\mathrm{A}^{\frac{kp}{kq}} = [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}]^{kp}$ [I. § 10 df. fract. pwr.

$= 1 \divideontimes [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}] \divideontimes [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}] \divideontimes \cdots kp'$ times

$= 1 \divideontimes [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}]^{k} \divideontimes [\sqrt[k]{(\sqrt[q]{\mathrm{A}})}]^{k} \divideontimes \cdots$ p' times

$= 1 \divideontimes (\sqrt[q]{\mathrm{A}}) \divideontimes (\sqrt[q]{\mathrm{A}}) \divideontimes \cdots$ p' times

$= \mathrm{A}^{\pm\frac{p'}{q}} = \mathrm{A}^{\frac{p}{q}}$,

$\therefore$ every value of either member is a value of the other. Q.E.D.

NOTE 1. In general, when either member of an equation admits more than one value, the sign of equality asserts that every value of either member is a value of the other.

NOTE 2. In what follows ${}^{+}\mathrm{A}^{\frac{p}{q}} \equiv$ the positive value of $\sqrt[q]{\mathrm{A}^{p}}$.

PRODUCT OF LIKE POWERS.

THEOR. 2. *Every value of the product of like commensurable powers of two or more bases is a value of the like power of their product.*

Let A, B, C, ⋯ be any bases, and A^n, B^n, C^n, ⋯ like commensurable powers of them;

then will every value of $A^n \cdot B^n \cdot C^n \cdots$ be a value of $\overline{A \cdot B \cdot C \cdots}^n$.

(*a*) n *an integer, positive or negative.* [II. th. 3 cr. 12

(*b*) n *a simple fraction.*

For let $n \equiv \frac{p}{q}$, wherein q is a positive integer, p an integer either positive or negative;

$$\text{then} \because A^{\frac{p}{q}} = (A^{\frac{1}{q}})^p, \quad B^{\frac{p}{q}} = (B^{\frac{1}{q}})^p, \quad C^{\frac{p}{q}} = (C^{\frac{1}{q}})^p, \cdots \quad \text{[df. fract. pwr.}$$

$$\therefore A^{\frac{p}{q}} \cdot B^{\frac{p}{q}} \cdot C^{\frac{p}{q}} \cdots = (A^{\frac{1}{q}})^p \cdot (B^{\frac{1}{q}})^p \cdot (C^{\frac{1}{q}})^p \cdots$$

$$= (A^{\frac{1}{q}} \cdot B^{\frac{1}{q}} \cdot C^{\frac{1}{q}} \cdots)^p. \quad [(a)$$

$$\text{But} \because (A^{\frac{1}{q}} \cdot B^{\frac{1}{q}} \cdot C^{\frac{1}{q}} \cdots)^q = (A^{\frac{1}{q}})^q \cdot (B^{\frac{1}{q}})^q \cdot (C^{\frac{1}{q}})^q \cdots \quad [(a)$$

$$= A \cdot B \cdot C \cdots,$$

$\therefore$ every value of $A^{\frac{1}{q}} \cdot B^{\frac{1}{q}} \cdot C^{\frac{1}{q}} \cdots$ is a value of $(A \cdot B \cdot C \cdots)^{\frac{1}{q}}$; [df. root

$\therefore$ every value of $A^{\frac{p}{q}} \cdot B^{\frac{p}{q}} \cdot C^{\frac{p}{q}} \cdots$, $= (A^{\frac{1}{q}} \cdot B^{\frac{1}{q}} \cdot C^{\frac{1}{q}} \cdots)^p$, [above

is a value of $(A \cdot B \cdot C \cdots)^{\frac{p}{q}}$. Q.E.D.

NOTE 1. In the demonstration of case (*b*) nothing need be said of the series to which the powers $A^{\frac{p}{q}}$, $B^{\frac{p}{q}}$, ⋯ belong; for the demonstration holds, and the theorem is true, whichever values of the roots of A, B, ⋯ be taken.

E.g., of $4^{\frac{3}{2}}$, $9^{\frac{3}{2}}$, $36^{\frac{3}{2}}$, the values are $^{\pm}8$, $^{\pm}27$, $^{\pm}216$,
whereof the product $^{+}8 \cdot {}^{+}27$, or $^{-}8 \cdot {}^{-}27$, is $^{+}216$,
and the product $^{-}8 \cdot {}^{+}27$, or $^{+}8 \cdot {}^{-}27$, is $^{-}216$.

NOTE 2. When the exponent is fractional and some of the bases are alike or so related that their powers must be in the same series, the product of the powers may admit fewer values than the power of the product.

E.g., $\sqrt{a} \cdot \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{4b^3} = +2ab^2$ only,
but $\sqrt{4a^2b^4} = \pm 2ab^2$.

COR. *Every value of the quotient of like commensurable powers of two bases is a value of the like power of their quotient.*

E.g., $\sqrt{a}:\sqrt{b}=\pm\sqrt{(a:b)}$; $\sqrt{a}:\sqrt{4a}=+1:\sqrt{4}={}^{\pm}\frac{1}{2}$.

PRODUCT OF POWERS OF SAME BASE.

THEOR. 3. *The product of two or more commensurable powers of any same base, in any same series, is that power of the base whose exponent is the sum of the exponents of the factors.*

Let A^m, A^n, $\cdots$ be any commensurable powers of a base A, in the same series;

then will $A^m \cdot A^n \cdots = A^{m+n+\cdots}$.

(*a*) m, n, $\cdots$, *all integers, whether positive or negative.* [II. th. 3 cr. 10

(*b*) m, n, $\cdots$, *some or all of them simple fractions.*

For let $m \equiv \frac{p}{q}$, $n \equiv \frac{r}{s}$, $\cdots$, wherein $p, q, r, s, \cdots$ are all integers, and the denominators $q, s, \cdots$ are all positive;

and let k be the l. c. mlt. of $q, s, \cdots$, so that $\frac{p}{q}=\frac{g}{k}$, $\frac{r}{s}=\frac{h}{k}$, $\cdots$;

then $\because$ $A^{\frac{p}{q}}=A^{\frac{g}{k}}=(A^{\frac{1}{k}})^g$, $A^{\frac{r}{s}}=A^{\frac{h}{k}}=(A^{\frac{1}{k}})^h\cdots$, [th. 1, df. fract. pwr.

wherein $A^{\frac{1}{k}}$ preserves throughout the same value, [same series

$$\begin{aligned}
\therefore\ A^{\frac{p}{q}} \cdot A^{\frac{r}{s}} \cdots &= (A^{\frac{1}{k}})^g \cdot (A^{\frac{1}{k}})^h \cdots \\
&= (A^{\frac{1}{k}})^{g+h+\cdots} && [(a) \\
&= A^{\frac{g+h+\cdots}{k}} && [\text{df. fract. pwr.} \\
&= A^{\frac{g}{k}+\frac{h}{k}+\cdots},
\end{aligned}$$

i.e., $A^m \cdot A^n \cdots = A^{m+n+\cdots}$. Q.E.D.

COR. 1. *The quotient of two commensurable powers of any same base, in any same series, is that power of the base whose exponent is the excess of the exponent of the dividend over that of the divisor.*

COR. 2. *Of two or more commensurable powers of any same base, in any same series, the product or quotient is in the same series.*

POWER OF A POWER.

THEOR. 4. *A commensurable power of a commensurable power of any base equals, or includes among its values, that power of the base whose exponent is the product of the given exponents.*

Let A be any base, and m, n any commensurable numbers; then will $(A^m)^n$ have every value of A^{mn}.

(*a*) n *a positive integer.*

For $(A^m)^n = 1 \cdot A^m \cdot A^m \cdots$, n times [I. § 10 df. int. pwr.

$= A^{(m+m+\cdots,\ n\text{ times})}$ [th. 3

$= A^{mn}$. Q.E.D.

(*b*) n *a negative integer.*

For $\because$ $-n$ is then a positive integer,

$\therefore$ $(A^m)^n = 1 : A^m : A^m : \cdots -n$ times [df. int. pwr.

$= 1 : (A^m \cdot A^m \cdot \cdots -n$ times) [II. th. 3 cr. 8

$= 1 : A^{m(-n)}$ [th. 3

$= 1 : A^{-mn}$

$= A^{mn}$. Q.E.D. [df. commens. pwr.

(*c*) n *a fraction* $\dfrac{p}{q}$; p, q *integers*, q *positive.*

For $\because$ $(A^{\frac{m}{q}})^q = A^m$, [(*a*)

$\therefore$ every value of $A^{\frac{m}{q}}$ is a value of $\sqrt[q]{(A^m)}$, [df. root

$\therefore$ every value of $A^{m \cdot \frac{p}{q}}$, $= A^{\frac{m}{q} \cdot p}$, $= (A^{\frac{m}{q}})^p$, [(*a*, *b*)

is a value of $[\sqrt[q]{(A^m)}]^p$,

i.e., it is a value of $(A^m)^{\frac{p}{q}}$, [df. fract. pwr.

$\therefore$ every value of A^{mn} is a value of $(A^m)^n$. Q.E.D.

COR. 1. *If* m, n *be commensurable numbers*, $(A^m)^n$ *and* $(A^n)^m$ *have at least one value in common, viz.*, A^{mn}.

COR. 2. *In particular, if* $B = A^{\pm\frac{p}{q}}$, *then* $B^{\pm\frac{q}{p}}$ *has the value* A *whether or not it have other values also.*

COR. 3. *The reciprocal of any power of a base is the like power of the reciprocal of the base;*

i.e., $1 : A^{\pm\frac{p}{q}} = 1 \dot{\times} A^{\frac{1}{q}} \dot{\times} A^{\frac{1}{q}} \dot{\times} \cdots p$ times, [II. th. 3 cr. 8

$$= (1 : A)^{\pm\frac{p}{q}} \quad \text{or} \quad (A^{\pm\frac{p}{q}})^{-1} = (A^{-1})^{\pm\frac{p}{q}}.$$

NOTE. If n be a fraction $\frac{p}{q}$, and q be not prime to m, or to its numerator if m be a fraction, then $(A^m)^n$ may have values not included among those of A^{mn}.

E.g., $(10^2)^{\frac{3}{2}}$ has not alone the value $10^3 = {}^+1000$;
but also $(10^2)^{\frac{3}{2}} = (100)^{\frac{3}{2}} = (100^{\frac{1}{2}})^3 = ({}^\pm 10)^3 = {}^\pm 1000$.

So, $(\sqrt{x})^4 = x^{\frac{4}{2}} = +x^2$ only;
but $\sqrt{x^4} = (x^4)^{\frac{1}{2}} = \pm x^2$.

§ 3. CONTINUITY OF COMMENSURABLE POWERS.

THEOR. 5. *If there be a positive base* A $\left\{\begin{matrix}\textit{larger}\\ \textit{smaller}\end{matrix}\right.$ *than unity, then:*

1. *For every positive commensurable exponent* n *the power* A^n *has one positive value* ${}^+A^n \left\{\begin{matrix}\textit{larger}\\ \textit{smaller}\end{matrix}\right.$ *than unity;*

2. *For every negative commensurable exponent* n *the power* A^n *has one positive value* ${}^+A^n \left\{\begin{matrix}\textit{smaller}\\ \textit{larger}\end{matrix}\right.$ *than unity;*

3. *In either case,* A^n *has but one such value.*

1. n *positive.*

(*a*) n *a positive integer.*

For $\because A^n = 1 \cdot A \cdot A \cdots$, n times,
and $\because$ each of the factors A is positive and $\left\{ \gtrless 1 \right.$,
$\therefore$ the product A^n is positive and $\left\{ \gtrless 1 \right.$. Q.E.D. [II. § 3 ax. 19

(*b*) n *the reciprocal of a positive integer* q.

For let x, a variable, $\left\{\begin{matrix}\text{increase}\\ \text{decrease}\end{matrix}\right.$ continuously from the value 1 to the value A;
then $\because$ inc x^q : inc $x \doteq$ deriv. x^q, $= qx^{q-1}$, as inc $x \doteq 0$, [VII. th. 17
$\therefore$ when inc x is infinitesimal, so is inc x^q; [VII. th. 2 cr. 1
i.e., as x passes continuously through all values from 1 to A, x^q takes every value between 1 and A^q;
but $\because A^q$, $= 1 \cdot A \cdot A \cdots$, q times, $\left\{ \gtrless A \right.$ when $A \left\{ \gtrless 1 \right.$,
$\therefore$ A is a value between 1 and A^q;
$\therefore$ x^q passes through the value A,
and $A^{\frac{1}{q}}$ has the value x, a positive number $\left\{ \gtrless 1 \right.$. Q.E.D.

(*c*) n *any positive fraction* $\frac{p}{q}$.

For $\because$ $A^{\frac{1}{q}}$ has a positive value $\{ \gtrless 1$, [(*b*)

$\therefore$ $A^{\frac{p}{q}}$, $=(A^{\frac{1}{q}})^p$, has a positive value $\{ \gtrless 1$. [Q.E.D. [(*a*)

NOTE. $A^{\frac{p}{q}}$ is not necessarily commensurable, even when A is commensurable.

2. n *negative*.

Let n, the exponent, $=-m$, wherein m is positive;

then $\because$ A^m has a positive value $^+A^m$, $\{ \gtrless 1$, [(1)

$\therefore$ A^n, $=1 : A^m$, has a positive value $^+A^n$, $\{ \lessgtr 1$. Q.E.D.

3. *But one positive value.*

For if possible let the root $A^{\frac{1}{q}}$ have two positive values unequally large;

then the product A, $=1 \cdot A^{\frac{1}{q}} \cdot A^{\frac{1}{q}} \cdots$ q times, has two values unequally large, [II. § 3, ax.19

which is contrary to the hypothesis;

$\therefore$ $A^{\frac{1}{q}}$ has but one positive value,

$\therefore$ the product $A^{\frac{p}{q}}$, $=1 \cdot A^{\frac{1}{q}} \cdot A^{\frac{1}{q}} \cdots$ p times, has but one positive value.

So with $A^{-\frac{p}{q}}$, $=1 : A^{\frac{p}{q}}$.

COR. *If the base and exponent be both finite, so is the positive value of the power.*

(*a*) *The exponent an integer, either positive or negative.*

For $\because$ the power is the continuous $\{$ product / quotient of 1 by a finite number of finite $\{$ multipliers, / divisors, [df. int. pwr.

$\therefore$ the power is finite. Q.E.D. [VII. th.11 cr.3

(*b*) *The exponent a fraction.*

Let the base $A \{ \gtrless 1$, and let the exponent n lie between the integers i and $i+1$;

then $\because$ $n-i$ and $i+1-n$ are positive and commensurable,

$\therefore$ A^{n-i}, A^{i+1-n}, both $\{ \gtrless 1$; [th.5

and $\because$ $A^n = A^{n-i} \cdot A^i$, $A^{i+1} = A^{i+1-n} \cdot A^n$, [th. 3

$\therefore$ $A^n \left\{ \begin{matrix} > \\ < \end{matrix} \right. A^i$, $A^{i+1} \left\{ \begin{matrix} > \\ < \end{matrix} \right. A^n$,

i.e., ${}^+A^n$ lies between A^i and A^{i+1}, which are both finite and positive. [(*a*)

$\therefore$ ${}^+A^n$ is finite. Q.E.D.

THEOR. 6. *Of a commensurable* $\left\{ \begin{matrix} \textit{positive} \\ \textit{negative} \end{matrix} \right.$ *power of a variable positive base with given exponent, the positive value is* $\left\{ \begin{matrix} \textit{an increasing} \\ \textit{a decreasing} \end{matrix} \right.$ *continuous function of the base.*

Let A be the variable base, and $n, = \pm\frac{p}{q}$, the given $\left\{ \begin{matrix} \text{positive} \\ \text{negative} \end{matrix} \right.$ exponent; then:

(*a*) *The larger the base* A, *the* $\left\{ \begin{matrix} \textit{larger} \\ \textit{smaller} \end{matrix} \right.$ *the power.*

For $\because$ $A = 1 \cdot A^{\frac{1}{q}} \cdot A^{\frac{1}{q}} \cdots q$ times,

$\therefore$ the larger $A^{\frac{1}{q}}$ is, the larger is A; [II. § 3 ax. 19

i.e., the larger A is, the larger is $A^{\frac{1}{q}}$;

and $\because$ $A^{\frac{p}{q}} = 1 \cdot A^{\frac{1}{q}} \cdot A^{\frac{1}{q}} \cdots p$ times,

$\therefore$ the larger $A^{\frac{1}{q}}$ is, the larger is $A^{\frac{p}{q}}$;

$\therefore$ the larger A is, the larger is $A^{\frac{p}{q}}$;

and the smaller is $A^{-\frac{p}{q}}, = 1 : {}^+A^{\frac{p}{q}}$;

i.e., when A increases, $A^{+\frac{p}{q}}$ increases, but $A^{-\frac{p}{q}}$ decreases. Q.E.D.

(*b*) *When* A *passes through every value from* 0 *to* ${}^+\infty$ *in order,* ${}^+A^n$ *passes through every value from* $\left\{ \begin{matrix} 0 \;\textit{to}\; {}^+\infty \\ {}^+\infty \;\textit{to}\; 0 \end{matrix} \right.$ *in order.*

For, let B be any positive number;

then $\because$ the power $B^{\frac{1}{n}}$ has a positive value ${}^+B^{\frac{1}{n}}$, [th. 5

$\therefore$ if $A = {}^+B^{\frac{1}{n}}$, ${}^+A^n, = {}^+({}^+B^{\frac{1}{n}})^n$, takes the value B, [th. 4

$\therefore$ every number B from 0 to ${}^+\infty$ becomes in turn a value of ${}^+A^n$;

and $\because$ the larger the base, the $\left\{ \begin{matrix} \text{larger} \\ \text{smaller} \end{matrix} \right.$ the power, [(*a*)

$\therefore$ when A passes through every value from 0 to ${}^+\infty$ in order, ${}^+A^n$ passes through every value from $\left\{ \begin{matrix} 0 \text{ to } {}^+\infty \\ {}^+\infty \text{ to } 0 \end{matrix} \right.$ in order, *i.e.*, it is continuous. Q.E.D.

COR. 1. *If the base and exponent be both finite, every infinitesimal change in the base gives an infinitesimal change in the positive value of the power, and conversely.*

COR. 2. *If the base approach a limit* A_0, *the positive value of the power approaches a limit* ${}^+A_0{}^n$, *and conversely.*

THEOR. 7. *Of a variable commensurable power of a constant positive base larger than unity:*

1. *The positive value is an increasing function of the exponent;*

2. *The exponent can be so taken that the power shall lie between any two positive numbers, however close together.*

Let A, >1, be the base; and let n', n'' be any two values of the exponent n, such that $n' < n''$; then:

1. ${}^+A^n$ *is an increasing function of* n.

For $\because\ {}^+A^{n''} = {}^+A^{n'} \cdot {}^+A^{n''-n'}$ [th. 3

$= {}^+A^{n'} \cdot$ a positive power of A

$= {}^+A^{n'} \cdot$ a number greater than 1, [th. 5(1)

$\therefore\ {}^+A^{n''} > {}^+A^{n'}$;

i.e., ${}^+A^n$ increases with n. Q.E.D.

2. n *can be so taken that* ${}^+A^n$ *shall lie between any positive numbers* B, C, whereof $B < C$.

Take A', Q any positive integers so great that $A' > A$, $\frac{1}{Q} < \frac{C}{B} - 1$;

and let $h \equiv \frac{1}{A'Q}$;

then $\because\ \left(1 + \frac{1}{Q}\right)^{A'Q} = 1 + A'Q \cdot \frac{1}{Q} +$ other positive terms [bin. th.

$> 1 + A'$

$> A$,

$\therefore\ A^h \quad < 1 + \frac{1}{Q}$ [th. 6(a)

$< \frac{C}{B}$;

$\therefore$ of the series $\cdots {}^+A^{-3h}, {}^+A^{-2h}, {}^+A^{-h}, {}^+A^0, {}^+A^h, {}^+A^{2h}, {}^+A^{3h}$, $\cdots$ each term is less than the $\frac{C}{B}$th part of the term next before it.

But this series has terms $> \text{B}$;

for, if mh be any integer $> \dfrac{\text{B}}{\text{A}-1}$,

then $\text{A}^{mh}, = (1+\overline{\text{A}-1})^{mh}, > 1+mh(\text{A}-1) > \text{B}$. [bin. th.

And the series has terms $< \text{B}$;

for, if $\text{A}^{m'h}$ be any term $> \dfrac{1}{\text{B}}$,

then $\text{A}^{-m'h}, = 1 : \text{A}^{m'h}$, is a term $< \text{B}$.

Let A^{gh} be the greatest term of the series less than B;

then the next term, $\text{A}^{(g+1)h}, < \dfrac{\text{C}}{\text{B}} \cdot \text{B}$ or C, [above

i.e., $\text{A}^{(g+1)h}$, a commensurable power of A, lies between B and C. Q.E.D.

COR. 1. *Of a variable commensurable power of a constant positive base smaller than unity, the positive value is a decreasing function of the exponent, and can be made to lie between any two positive numbers.*

Let the base be $a, = 1 : \text{A}$, wherein $\text{A} > 1$; and let $b, = 1 : \text{B}$, and $c, = 1 : \text{C}$, be any two positive numbers;

then $\because$ $^+\text{A}^n$ increases with the exponent, and takes values between the positive numbers B, C, [th. 7

and $\because$ $^+a^n$ is the reciprocal of $^+\text{A}^n$, [th. 4 cr. 3

$\therefore$ $^+a^n$ decreases as the exponent increases, [II. ax. 18

and takes values between b and c. Q.E.D.

COR. 2. *When the base differs sensibly from* 0, 1, *and* ∞, *and the exponent is not* ∞, *then every infinitesimal change in the exponent gives an infinitesimal change in the positive value of the power, and conversely.*

COR. 3. *When the base differs sensibly from* 0, 1, *and* ∞, *and the exponent approaches a limit* n_0, *the positive value of the power approaches a limit* $^+\text{A}^{\text{n}_0}$; *and conversely.*

NOTE. The principles established in theors. 5, 6, 7 are summarized as follows:

Every commensurable power of a positive base has one and but one real positive value [th. 5]; $\left\{\begin{array}{l}\textit{an increasing}\\ \textit{a decreasing}\end{array}\right.$ *continuous function of the base if the base vary and the exponent be constant and* $\left\{\begin{array}{l}\textit{positive}\\ \textit{negative}\end{array}\right.$ [th. 6]; $\left\{\begin{array}{l}\textit{an increasing}\\ \textit{a decreasing}\end{array}\right.$ *function of the exponent if the exponent vary and the base be constant and* $\left\{\begin{array}{l}\textit{larger}\\ \textit{smaller}\end{array}\right.$ *than unity* [th. 7, th. 7 cr. 1].

Whether the base or the exponent varies, the commensurable power takes values [indeed, an infinite number of them] *between any two positive numbers however close together* [above, th. 7, cr. 1].

When both base and exponent are finite, and the base $\neq 1$, *any infinitesimal change in either gives an infinitesimal change in the positive value of the power, and conversely* [th. 6 cr. 1, th. 7 cr. 2]; *and if either the base or the exponent approach a limit while the other is constant, so does the positive value of the power, and conversely* [th. 6 cr. 2, th. 7 cr. 3].

The positive value of the power is finite when the base and the exponent are finite [th. 5 cr.]. *This value is* $\left\{\begin{array}{l}\textit{larger}\\ \textit{smaller}\end{array}\right.$ *than unity if the exponent be positive, and* $\left\{\begin{array}{l}\textit{smaller}\\ \textit{larger}\end{array}\right.$ *than unity if the exponent be negative, when the base is* $\left\{\begin{array}{l}\textit{larger}\\ \textit{smaller}\end{array}\right.$ *than unity* [th. 5].

It appears later [th. 12] that the powers of a constant base take a continuous series of values when the exponent takes a continuous series. But when the varying exponent or base is restricted to commensurable values, then between any two values taken by the power there lie an infinite number of values not so taken.

§ 4. INCOMMENSURABLE POWERS.

HITHERTO no meaning has been given to the symbol A^n when n is incommensurable; and any meaning that may now be given to it should fulfil, if possible, the following conditions:

1. It should give a single definite positive value to the symbol A^n when A has a given positive value and n is incommensurable.

2. It should not conflict with any use that the symbol A^n has when n is commensurable.

3. It should preserve all the fundamental properties that the symbol A^n has when n is commensurable: in particular, theorems 2, 3, 4 should be true for all real exponents whatever.

The following theorem lays a foundation for the definition:

THEOR. 8. *If there be a constant positive base not* 0 *nor* 1 *nor* ∞, *and two variable commensurable exponents, one increasing and the other decreasing toward a common incommensurable limit not* ∞, *then:*

1. *The positive values of the two variable commensurable powers have a common limit, which lies between them and is not* 0 *nor* ∞.

2. *This common limit depends upon the value of the base, and of the common limit of the exponents, but not upon the law by which either exponent approaches its limit.*

3. *This common limit is not a commensurable power of the base.*

Let A be any constant positive base not 0 nor 1 nor ∞; let x, y be any variable commensurable exponents, x increasing and y decreasing toward a common incommensurable limit n not infinite; and let x', y' stand respectively for a particular series of values of x, y that approach n as their common limit, and so with x'', y'', with x''', y''', $\cdots$, then:

1. ${}^+A^{x'}$, ${}^+A^{y'}$ *have a common limit that lies between them and is not* 0 *nor* ∞.

For $\because$ the exponents x', y' each $\doteq n$, their common limit, [hyp.

$\therefore$ x', y' come to differ from each other by less than any assigned number,

$\therefore$ ${}^+A^{x'}$, ${}^+A^{y'}$ come to differ from each other by less than any assigned number; [th. 7 cr. 2

and $\because$ $x' < y'$ always, and x' increases while y' decreases,

$\therefore$ ${}^+A^{x'} \left\{ \begin{matrix} < \\ > \end{matrix} \right. {}^+A^{y'}$ always, and ${}^+A^{x'} \left\{ \begin{matrix} \text{increases} \\ \text{decreases} \end{matrix} \right.$ while ${}^+A^{y'}$ $\left\{ \begin{matrix} \text{decreases,} \\ \text{increases,} \end{matrix} \right.$ when $A \left\{ \begin{matrix} > \\ < \end{matrix} \right. 1.$ [th.7(1), th.7 cr.1

$\therefore$ ${}^+A^{x'}$, ${}^+A^{y'}$ approach a common limit that lies between them, and therefore is not 0 nor ∞. Q.E.D. [VII. th.1

2. *The variable powers* ${}^+A^{x'}$, ${}^+A^{y'}$, ${}^+A^{x''}$, ${}^+A^{y''}$, $\cdots$ *have the same common limit.*

For the variable powers ${}^+A^{x''}$, ${}^+A^{y'}$ have a common limit, the same as the limit of ${}^+A^{y'}$,

i.e., the same as the common limit of ${}^+A^{x'}$, ${}^+A^{y'}$.

So the variable powers ${}^+A^{x''}$, ${}^+A^{y''}$ have a common limit, the same as the limit of ${}^+A^{x''}$, ${}^+A^{y'}$; and so on.

3. *This common limit of* ${}^+A^{x}$, ${}^+A^{y}$ *is not a commensurable power of* A.

For, if possible, let this limit be some commensurable power, ${}^+A^{m}$; then is the commensurable exponent m the common limit of the variable commensurable exponents x, y, [th.7 cr.3 cnv.

which is contrary to the hypothesis;

$\therefore$ this supposition fails, and it is only left that the common limit of ${}^+A^{x}$, ${}^+A^{y}$, $\cdots$ be not a commensurable power of A. Q.E.D.

DEFINITION.

If there be two variable commensurable powers of a given base, the one increasing and the other decreasing, and such that their variable exponents have a common incommensurable limit that lies between them, then the symbol formed by writing the base with this limit for exponent stands for the common limit of the positive values of the variable powers and is an *incommensurable power* of the base.

That this definition satisfies the first two of the conditions stated above is evident from theor. 7; and that it satisfies the third condition appears from the theorems that follow.

NOTE. It appears later that A^n may have all the limiting values of A^x, A^y, *i.e.*, every value of $A^x \doteq$ some value of A^n; but only the positive limiting values are considered here.

§ 5. COMBINATIONS OF POWERS IN GENERAL.

PRODUCT OF LIKE POWERS.

THEOR. 9. *Every value of the product of like powers of two or more bases is a value of the like power of their product.*

Let n be any number and A, B, C, ⋯ be any bases;
then is every value of $A^n \cdot B^n \cdot C^n \cdots$ a value of $\overline{A \cdot B \cdot C \cdots}^n$.

(*a*, *b*) n *commensurable.* [th. 2

(*c*) n *incommensurable.*

For let x be a commensurable variable whose limit is n;

then $\because$ $A^x \doteq A^n$, $B^x \doteq B^n$, $C^x \doteq C^n$, ⋯, [df. incom. pwr.

$\therefore$ $A^x \cdot B^x \cdot C^x \cdots \doteq A^n \cdot B^n \cdot C^n \cdots$; [VII. th. 8

and $\because$ $A^x \cdot B^x \cdot C^x \cdots =$ a value of $\overline{A \cdot B \cdot C \cdots}^x$ [th. 2

$\doteq \overline{A \cdot B \cdot C \cdots}^n$, [df. incom. pwr.

$\therefore$ $A^n \cdot B^n \cdot C^n \cdots =$ a value of $\overline{A \cdot B \cdot C \cdots}^n$. Q.E.D. [VII. th. 6 cr.

COR. *Every value of the quotient of like powers of two bases is a value of the like power of their quotient.*

PRODUCT OF POWERS OF SAME BASE.

THEOR. 10. *The product of two or more powers of any same base, in any same series, is that power of the base whose exponent is the sum of the exponents of the factors, and in the same series.*

Let m, n, ⋯ be any numbers, and A any base;
then will $A^m \cdot A^n \cdots = A^{m+n+\cdots}$.

(*a*, *b*) m, n, ⋯ *all commensurables.* [th. 3, cr. 2

(*c*) m, n, ⋯ *some or all of them incommensurable.*

For let x, y, ⋯ be commensurable variables whose limits are m, n, ⋯ respectively,

then $\because$ $A^x \doteq A^m$, $A^y \doteq A^n$, ⋯, [df. incom. pwr.

$\therefore$ $A^x \cdot A^y \cdots \doteq A^m \cdot A^n \cdots$. [VII. th. 8

But $\because$ $A^x \cdot A^y \cdots = A^{x+y+\cdots} \doteq A^{m+n+\cdots}$, [th. 3, df. incom. pwr.

$\therefore$ $A^m \cdot A^n \cdots = A^{m+n+\cdots}$. Q.E.D. [VII. th. 6 cr.

COR. *The quotient of two powers of any same base in any same series is that power of the base whose exponent is the excess of the exponent of the dividend over that of the divisor.*

POWER OF A POWER.

THEOR. 11. *A power of a power of any base equals, or includes among its values, that power of the base whose exponent is the product of the given exponents.*

Let m, n be any numbers, and A any base;
then will $(A^m)^n$ have every value of $A^{m \cdot n}$.

(a, b, c) m, n *commensurables.* [th. 4

(d) m, *or* n, *or both, incommensurables.*

For let x, y be commensurable variables whose limits are m, n:

then $A^x \doteq A^m$, [df. incom. pwr.

i.e., every value of $A^x \doteq$ some value of A^m, [nt. to df. inc. pwr.

$\therefore (A^x)^y \doteq (A^m)^y$ as $x \doteq m$;

but $(A^m)^y \doteq (A^m)^n$ as $y \doteq n$, [df.

$\therefore (A^x)^y \doteq (A^m)^n$ as $x \doteq m$ and $y \doteq n$;

and $\because A^{xy} \doteq A^{mn}$, [$xy \doteq mn$,

and $\because (A^x)^y$ equals, or includes among its values, A^{xy},

$\therefore (A^m)^n$ equals, or includes among its values, A^{mn}. Q.E.D.

So, when only one exponent, m or n, is incommensurable. Q.E.D.

COR. *Whatever the values of the exponents* m, n, *the powers* $(A^m)^n$ *and* $(A^n)^m$ *have at least one value in common, viz.,* A^{mn}.

NOTE. Most of §§ 2, 5, with some obvious results, may be summarized thus:

The values of any commensurable power $A^{n'}$ *depend upon* A *and the value, not the form, of* n'; *so with any incommensurable power* A^n, $= lim\, A^{n'}$ *as* $n \doteq n'$.

Any product or quotient of like powers, whether in one series or not, is the like power of the product or quotient of the bases; except that if the bases be not independent, the power of the product may, though rarely, have more values than the product of the powers.

Any product or quotient of powers of one base, in one series, is that power whose exponent is the sum or difference of the given exponents, and is in the same series.

Any power of a power is that power of the base whose exponent is the product of the given exponents; except that $(A^{\frac{m}{n}})^{\frac{p}{q}}$ *may, though rarely, have more values than* $A^{\frac{mp}{nq}}$, *if* q *be not prime to* m.

§ 6. CONTINUITY OF POWERS IN GENERAL.

LEMMA. *The positive value of any* $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right.$ *power of a positive base:*

1. *Is* $\left\{\begin{matrix}\textit{larger}\\ \textit{smaller}\end{matrix}\right.$ *than unity if the base be larger than unity.*

2. *Is* $\left\{\begin{matrix}\textit{smaller}\\ \textit{larger}\end{matrix}\right.$ *than unity if the base be smaller than unity.*

1. Let A, > 1, be any positive base, and n any $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ exponent; then is the positive value of $A^n \left\{\begin{matrix}\text{larger}\\ \text{smaller}\end{matrix}\right.$ than 1.

(*a*) n *commensurable.* [th. 5

(*b*) n *incommensurable.*

Let n', n'' be any two commensurable variables, both $\left\{\begin{matrix}\text{positive,}\\ \text{negative,}\end{matrix}\right.$ approaching n as their common limit in such wise that always $n' < n < n''$:

then $\because$ $A > 1$ and n', n'' are both $\left\{\begin{matrix}\text{positive,}\\ \text{negative,}\end{matrix}\right.$ [hyp.

$\therefore$ of $A^{n'}$, $A^{n''}$, the positive values both $\left\{\begin{matrix}>\\ <\end{matrix}\right. 1$; [(*a*)

and $\because$ the positive value of A^n lies between them, [th. 8, df.

$\therefore$ the positive value of $A^n \left\{\begin{matrix}>\\ <\end{matrix}\right. 1$. Q.E.D.

2. Let $A < 1$, and let n be $\left\{\begin{matrix}\text{positive:}\\ \text{negative:}\end{matrix}\right.$

then $\because$ $\frac{1}{A} > 1$,

$\therefore$ of A^n, $= \left(\frac{1}{A}\right)^{-n}$,

the positive value $\left\{\begin{matrix}<\\ >\end{matrix}\right. 1$ for $-n \left\{\begin{matrix}\text{negative,}\\ \text{positive,}\end{matrix}\right.$ [1

i.e., for $n \left\{\begin{matrix}\text{positive.}\\ \text{negative.}\end{matrix}\right.$ Q.E.D.

COR. *If* $A^n = 1$, *then either* $A = 1$ *or* $n = 0$.

For if neither $A = 1$ nor $n = 0$,

then is A^n larger or smaller than unity,

which is contrary to the hypothesis;

$\therefore$ either $A = 1$ or $n = 0$. Q.E.D.

THEOR. 12. *If there be a variable positive base* A, *and a constant* $\left\{\begin{matrix}\textit{positive}\\ \textit{negative}\end{matrix}\right.$ *exponent* n, *then:*

1. *To each value of the base there corresponds one and but one positive value of the power;* $\left\{\begin{matrix}\textit{an increasing}\\ \textit{a decreasing}\end{matrix}\right.$ *function of the base.*

2. *To each positive value of the power there corresponds one and but one positive value of the base;* $\left\{\begin{matrix}\textit{an increasing}\\ \textit{a decreasing}\end{matrix}\right.$ *function of the power.*

3. *The positive values of the power and of the base are continuous functions of each other.*

1. (*a*) n *commensurable.* [th. 5, 6

(*b*) n *incommensurable.*

For let A^n be the limit of a series of commensurable powers of A; then $\because$ each of these powers has one and but one positive value,

$\therefore$ A^n has one and but one positive value. Q.E.D.

So, let A', A'' be any two values of A, whereof $A' < A''$:

then $\because$ $A'' : A' > 1$,

$\therefore (A'' : A')^n \left\{\begin{matrix} > \\ < \end{matrix}\right. 1$; [lem.

$\therefore$ $A''^n, = (A'' : A')^n \cdot A'^n, \left\{\begin{matrix} > \\ < \end{matrix}\right. A'^n$,

i.e., the larger the base, the $\left\{\begin{matrix}\text{larger}\\ \text{smaller}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ power. Q.E.D.

2. Conversely:

$\because$ $A = (A^n)^{\frac{1}{n}}$,

$\therefore$ to each positive value of A^n there corresponds one and but one positive value of A; $\left\{\begin{matrix}\text{an increasing}\\ \text{a decreasing}\end{matrix}\right.$ function of A^n. Q.E.D. [1

3. Let A, always increasing, pass in order through every positive value from 0 to $^+\infty$:

then $\because$ A^n, $\left\{\begin{matrix}\text{an increasing}\\ \text{a decreasing}\end{matrix}\right.$ function of A, [1

takes in order every value

from $0^n, = \left\{\begin{matrix} 0 \\ \infty \end{matrix}\right.$, to $\infty^n, = \left\{\begin{matrix} \infty \\ 0 \end{matrix}\right.$, [2

$\therefore$ A^n is a continuous function of A. Q.E.D. [df.

So $A, = (A^n)^{\frac{1}{n}}$, is a continuous function of A^n. Q.E.D.

THEOR. 13. *If there be a constant positive base* A $\left\{ \begin{matrix} \textit{larger} \\ \textit{smaller} \end{matrix} \right.$ *than unity, and a variable exponent* n; *then:*

1. *To each value of the exponent there corresponds one and but one positive value of the power;* $\left\{ \begin{matrix} \textit{an increasing} \\ \textit{a decreasing} \end{matrix} \right.$ *function of the exponent.*

2. *To each positive value of the power there corresponds one and but one value of the exponent;* $\left\{ \begin{matrix} \textit{an increasing} \\ \textit{a decreasing} \end{matrix} \right.$ *function of the power.*

3. *The exponent and the positive value of the power are continuous functions of each other.*

1. For, when the base and exponent are given, there is one and but one positive value of the power. Q.E.D. [th.12.1

And this value is $\left\{ \begin{matrix} \text{an increasing} \\ \text{a decreasing} \end{matrix} \right.$ function of the exponent;

for let n_1, n_2 be any values of n whereof $n_2 > n_1$;

then ∵ $A^{n_2} = A^{n_2 - n_1} \cdot A^{n_1}$, [th. 10

and ∵ of $A^{n_2 - n_1}$ the positive value $\left\{ \begin{matrix} > \\ < \end{matrix} \right. 1$, [lemma; $n_2 - n_1 > 0$

∴ of A^{n_2} the positive value $\left\{ \begin{matrix} > \\ < \end{matrix} \right.$ that of A^{n_1}. Q.E.D. [II. ax.16

2. (*a*) *To each positive value* B *of* A^n *there corresponds one value of* n.

For let B′, B″ be any positive variables such that always B′ $\left\{ \begin{matrix} < \\ > \end{matrix} \right.$ B $\left\{ \begin{matrix} < \\ > \end{matrix} \right.$ B″, and approaching B as their common limit; and let variable commensurable exponents n', n'' be so taken that always the positive value of $A^{n'}$ shall lie between B′ and B, and the positive value of $A^{n''}$ shall lie between B and B″: [th. 7(2), cr. 1

then ∵ of the variable commensurable powers $A^{n'}$, $A^{n''}$ one increases and the other decreases toward B as their common limit, [hyp.

and the exponents n', n'' have a common limit n that lies between them, [th.7 cr. 3 cnv.

∴ the value of this common limit is a value of the exponent n corresponding to the value B of the power A^n. Q.E.D. [df. incom. pwr.

(*b*) *To each positive value* B *of* A^n *there corresponds but one value of* n.

For if A^m, A^n each $= B$,

then $\because A^{m-n} = A^m : A^n = B : B = 1$, [th.10 cr.

$\therefore m - n = 0$. [cr. to lem. th. 12

$\therefore m = n$. Q.E.D.

(*c*) *The exponent is* $\left\{ \begin{matrix} \textit{an increasing} \\ \textit{a decreasing} \end{matrix} \right.$ *function of the positive value of the power.*

For this is equivalent to the statement, already proved, that the positive value of the power is $\left\{ \begin{matrix} \text{an increasing} \\ \text{a decreasing} \end{matrix} \right.$ function of the exponent. [1

3. For to every value of the exponent there corresponds one and but one positive value of the power, and conversely; [1, 2

and $\because$ as the exponent increases the positive value of the power $\left\{ \begin{matrix} \text{always increases,} \\ \text{always decreases,} \end{matrix} \right.$ and conversely; [1, 2

$\therefore$ as the exponent passes in order through all values from $^-\infty$ to $^+\infty$, the positive value of the power passes in order through all values from $\left\{ \begin{matrix} 0 \text{ to } {}^+\infty, \\ {}^+\infty \text{ to } 0, \end{matrix} \right.$

and conversely, as the power passes from $\left\{ \begin{matrix} 0 \text{ to } {}^+\infty, \\ {}^+\infty \text{ to } 0, \end{matrix} \right.$ the exponent passes through all values from $^-\infty$ to $^+\infty$;

and $\because$ the power is the limit of a corresponding commensurable power that changes infinitesimally when the exponent changes infinitesimally, and conversely,

$\therefore$ every infinitesimal change in either the exponent or the power gives an infinitesimal change in the other,

$\therefore$ both exponent and power are continuous functions of each other. Q.E.D. [df. contin. func.

§ 7. DERIVATIVES OF POWERS.

DERIVATIVE OF A POWER OF A VARIABLE BASE.

THEOR. 14. *The derivative as to any variable base of a power of that base is the product of the given exponent into a power of the base whose exponent is a unit less than the given exponent.*

Let x be any variable and n any number;
then will $\text{D}_x x^n = n \cdot x^{n-1}$.

(a, b, c) n *commensurable.* [VII. th. 17

(d) n *incommensurable.*

For let n' be a commensurable variable independent of x and such that $n' \doteq n$, and let x take any increment h;

then $\because$ $x^{n'}$, x^n take the increments $(x+h)^{n'} - x^{n'}$, $(x+h)^n - x^n$,

and $\because$ $x^{n'} \doteq x^n$, $(x+h)^{n'} \doteq (x+h)^n$, as $n' \doteq n$, [df. incom. pwr.

$\therefore$ $(x+h)^{n'} - x^{n'} \doteq (x+h)^n - x^n$ as $n' \doteq n$, whatever h may be, [VII. th. 7

i.e., $\text{inc}\, x^{n'} \doteq \text{inc}\, x^n$

and $\dfrac{\text{inc}\, x^{n'}}{\text{inc}\, x} \doteq \dfrac{\text{inc}\, x^n}{\text{inc}\, x}$, as $n' \doteq n$, however small $\text{inc}\, x$,

$\therefore$ $\lim \dfrac{\text{inc}\, x^{n'}}{\text{inc}\, x} \doteq \lim \dfrac{\text{inc}\, x^n}{\text{inc}\, x}$ as $n' \doteq n$ and $\text{inc}\, x \doteq 0$,

i.e., $\text{D}_x x^{n'} \doteq \text{D}_x x^n$. [df. deriv.

But $\because$ $\text{D}_x x^{n'} = n' \cdot x^{n'-1}$, [VII. th. 17

and $\because$ $n' \cdot x^{n'-1} \doteq n \cdot x^{n-1}$ as $n' \doteq n$, [VII. th. 8

$\therefore$ $\text{D}_x x^n = n \cdot x^{n-1}$. Q.E.D. [VII. th. 6 cr.

DERIVATIVE OF A VARIABLE POWER.

THEOR. 15. *The derivative as to any variable of that power of a base whose exponent is the variable, is the quotient of the given power by a constant whose value depends upon the base alone.*

Let A be any base, M_A a certain function of that base, x any variable;
then will $\text{D}_x \text{A}^x = \text{A}^x : \text{M}_\text{A}$.

For let x take any increment h;

then $\because$ $\text{A}^{x+h} - \text{A}^x = \text{A}^x \cdot (\text{A}^h - 1)$, [th. 10

$\therefore\ \dfrac{A^{x+h}-A^x}{h} = A^x \cdot \dfrac{A^h-1}{h},$

$\therefore\ D_x A^x = A^x \cdot \lim \dfrac{A^h-1}{h}$ when $h \doteq 0.$

But $\lim \dfrac{A^h-1}{h} = \dfrac{D_h(A^h-1)}{D_h h}$ when $h \doteq 0,$ [VII. th. 18

$= D_h A^h$ when $h = 0,$

an expression free from x and a function of A only;

and $\because$ A^h has a single value for any one value of h, [th. 13 (1)

$\therefore\ \dfrac{A^h-1}{h}$ has a single value,

and $\lim \dfrac{A^h-1}{h}$, when $h \doteq 0$, has a single value, dependent on A alone.

Put $\dfrac{1}{M_A} \equiv \lim \dfrac{A^h-1}{h}$, when $h \doteq 0,$

then $D_x A^x = A^x : M_A.$ Q.E.D.

COR. *If* e *be such a number that* $M_e = 1$, *then* $D_x e^x = e^x$.

NOTE. The function M_A is called the *modulus* of that system of logarithms whose base is A; its value is found by methods in [XII. pr. 11]. The base e is the base of the Napierian system of logarithms.

DERIVATIVE OF A LOGARITHM.

THEOR. 16. *The derivative as to any variable of a logarithm of that variable is the quotient of the modulus of the system by the variable.*

Let x be any variable, A the base, and M_A the modulus of the system;

then will $D_x \log_A x = M_A : x$.

For put $y \equiv \log_A x$;

then $\because$ $x = A^y$, [I. § 11 df. log

$\therefore\ D_y x = D_y A^y = A^y : M_A.$ [th. 15

But $\because$ $D_x y = 1 : D_y x$

$= M_A : A^y,$

$\therefore\ D_x \log x = M_A : x.$ Q.E.D.

COR. $D_x log_e x = 1 : x.$

§ 8. RADICALS.

A *radical* is an indicated root of a number. There may be a coefficient; and then the whole expression is called a radical, and the indicated root is the *radical factor*.

A radical is $\left\{ \begin{matrix} \textit{rational} \\ \textit{irrational} \end{matrix} \right.$ if the root $\left\{ \begin{matrix} \text{can} \\ \text{cannot} \end{matrix} \right.$ be found and exactly expressed in commensurable real numbers [I. § 1], or in rational literal expressions [I. § 12]. Its value is $\left\{ \begin{matrix} \textit{real} \\ \textit{imaginary} \end{matrix} \right.$ if it $\left\{ \begin{matrix} \text{do not involve} \\ \text{involve} \end{matrix} \right.$ the even root of a negative.

E.g., $\sqrt[8]{256}$, $\sqrt[3]{8}$, $\sqrt[3]{-8}$, $\sqrt[5]{a^5}$, $\sqrt{(a^2+2ab+b^2)}$ are radicals that have the rational values

$^{\pm}2, \quad 2, \quad ^{-}2, \quad a, \quad \pm(a+b),$

besides certain irrational values discussed later.

But $\sqrt{x}$, $\sqrt{a^3}$, $\sqrt[3]{a^4}$, $\frac{4}{9}a \cdot a^{-\frac{1}{4}}$, $\frac{5}{2}(a^2+b^2)^{\frac{1}{2}}$ are irrational,

and $\sqrt{-1}$, $\sqrt{-a^2}$, $\sqrt[4]{-2a^3}$, $\frac{4}{9}a\cdot(-a)^{\frac{3}{4}}$, $\frac{5}{2}(a+b\sqrt{-1})^{\frac{2}{3}}$ are imaginaries; the first two of them commensurable, and the others not.

An expression that contains a radical is a *radical expression*.

A radical expression that cannot be freed from roots is an *irrational expression*, or *surd* [I. § 12].

An equation that contains surds is *rationalized* when it is replaced by an equivalent equation free from surds.

E.g., the equation $x=\sqrt{2}$, *i.e.*, $x=\sqrt[+]{2}$ or $\sqrt[-]{2}$, when rationalized, becomes $x^2=2$.

Roots of rational bases, and integral powers of such roots, with rational coefficients, if any, are *simple* radicals; and a radical is in its *simplest form* when its coefficient is real and entire, its exponent positive and less than unity, its root-index the smallest possible, and its base a real and entire expression containing no factor to a power whose degree equals or exceeds the root-index.

If a simple radical be surd, it is a *simple surd*.

The *degree* of a simple radical is the value of its root-index.

A simple radical is *quadratic*, *cubic*, *quartic* or *biquadratic*, ⋯ when the root-index is 2, 3, 4, ⋯.

E.g., $\frac{3}{2}(a^2+b^2)^{\frac{1}{2}}$, $3ab^2\cdot\sqrt[3]{(a^2-bc^3)}$, $a^2\cdot a^{\frac{3}{4}}$, $\sqrt[4]{-3}$,
are simple quadratic, cubic, and quartic surds in their simplest forms.

But $^{\pm}\sqrt{a^3}$, $\sqrt[3]{a^4}$, $\sqrt{8}$, $\sqrt{-8}$, $\frac{3}{2}(a^2c^2+b^2c^2)^{\frac{1}{2}}$, $\sqrt[3]{-\frac{5}{7}}$,
are simple radicals not in their simplest forms; for they may be severally reduced to:

$$^{\pm}a\sqrt{a},\ \ a\sqrt[3]{a},\ \ 2\sqrt{2},\ \ 2\sqrt{-2},\ \ \frac{3}{2}c\cdot(a^2+b^2)^{\frac{1}{2}},$$

$$\frac{^-1}{7}\sqrt[3]{245},\ =\frac{^-1}{7}\cdot 245^{\frac{1}{3}}.$$

And $\sqrt{[2-\sqrt[3]{(3+\sqrt[4]{4})}]}$, $(a^{\frac{3}{2}}+b^{\frac{3}{2}})^{\frac{2}{3}}$, $(a+b\sqrt{-1})^n$,
are complex radicals or surds.

Two radicals are *like*, or *similar*, if they have the same radical factor; $\left\{\begin{array}{l}\textit{conformable}\\ \textit{non-conformable}\end{array}\right.$ if they $\left\{\begin{array}{l}\text{can}\\ \text{cannot}\end{array}\right.$ be made like by transformation.

E.g., $3a^{\frac{1}{2}}$, $-5\sqrt{a}$; $2x\cdot(a^2+b^2)^{\frac{2}{3}}$, $-4y\cdot(a^2+b^2)^{\frac{2}{3}}$,
$8(x-y)\cdot(a^2+b^2)^{\frac{2}{3}}$, are like radicals,

and $\sqrt{18}$, $\sqrt{32}$, $\sqrt{98}$, are conformable.

The sum of two non-conformable simple surds, or of a rational expression and a simple surd, is a *binomial surd;* the sum of three non-conformable simple surds, or of two such surds and a rational expression, is *a trinomial surd;* and so on.

Two quadratic binomial surds are *conjugate* if they differ only in the sign of one term.

E.g., $a+\sqrt{b}$, $a-\sqrt{b}$; $10^{\frac{1}{2}}+3$, $10^{\frac{1}{2}}-3$;
$\sqrt{x}+\sqrt{(y+z)}$, $\sqrt{x}-\sqrt{(y+z)}$.

Two surds are *complementary* if their product be rational.

E.g., $a^{\frac{1}{3}}$, $a^{\frac{2}{3}}$; $5^{\frac{2}{7}}$, $5^{-\frac{2}{7}}$; $\sqrt{(a^2+b^2)}$, $\sqrt{(a^2+b^2)}$;
$a+\sqrt{b}$, $a-\sqrt{b}$; $a+b\sqrt{-1}$, $a-b\sqrt{-1}$.

So, any two conjugate binomial surds are complementary.

E.g., $a+\sqrt{b}$, $a-\sqrt{b}$; $2+3\sqrt{1}$, $2-3\sqrt{1}$.

THEOR. 17. *If two simple surds in their simplest form be equal, their coefficients are equal and their radical parts are equal.*

Let $a\sqrt[m]{A}$, $b\sqrt[n]{B}$ be equal simple surds in their simplest form; then will $a=b$, $A=B$, $m=n$.

For let $m=fp$, $n=fq$,

wherein f is the h.c.msr. of m, n, and p is prime to q;

then $\because\ a\cdot\sqrt[fp]{A} = b\cdot\sqrt[fq]{B}$, [hyp.

$\therefore\ a^{fp}\cdot A = b^{fp}\cdot(\sqrt[fq]{B})^{fp}$ [II. ax. 6

$= b^{fp}\cdot(\sqrt[q]{B})^{p}$; [df. fract. pwr., th. 1

$\therefore\ a^{fp}A : b^{fp} = (\sqrt[q]{B})^{p}$,

a true equation, but true only when $(\sqrt[q]{B})^p$ is rational,

i.e., when $p=q$ and $m=n$; Q.E.D.

$\therefore\ a:b = \sqrt[m]{(B:A)}$, [th. 2 cr. 1

a true equation, but true only when $\sqrt[m]{(B:A)}$ is rational,

i.e., when $A=B$ and $a=b$. Q.E.D.

COR. 1. *Two non-conformable surds cannot be equal.*

COR. 2. *The product or quotient of two* $\left\{\begin{array}{l}\textit{conformable}\\ \textit{non-conformable}\end{array}\right.$ *simple quadratic surds is* $\left\{\begin{array}{l}\textit{rational.}\\ \textit{surd.}\end{array}\right.$

E.g., $\sqrt{6}$ is conformable with $\sqrt{\frac{2}{3}}$ but not with $\sqrt{5}$,

and $\sqrt{(6\cdot\frac{2}{3})}$, $\sqrt{(6:\frac{2}{3})}$, $\sqrt{(\frac{2}{3}:6)}$ are the rationals
± 2, ± 3, $\pm\frac{1}{3}$,

but $\sqrt{(6\cdot 5)}$, $\sqrt{(6:5)}$, $\sqrt{(5:\frac{2}{3})}$, etc., are surds.

COR. 3. *If the continued product or quotient of two or more simple quadratic surds be* $\left\{\begin{array}{l}\textit{rational,}\\ \textit{surd,}\end{array}\right.$ *then the continued product or quotient of any of them, and the continued product or quotient of the rest of them, are* $\left\{\begin{array}{l}\textit{conformable.}\\ \textit{non-conformable.}\end{array}\right.$

E.g., $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{6}=6$; $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5}=\sqrt{30}$,

and $\sqrt{(2\cdot 3)}$, $\sqrt{6}$; $\sqrt{(2:3)}$, $\sqrt{6}$.; ⋯ are conformable;

but $\sqrt{(2\cdot 3)}$, $\sqrt{5}$; $\sqrt{(2:3)}$, $\sqrt{5}$; ⋯ are non-conformable.

THEOR. 18. *The sum of a finite number of simple non-conformable surds cannot be rational.*

Let $a_1\sqrt[p]{A_1}$, $a_2\sqrt[q]{A_2}$, $\cdots$ $a_n\sqrt[v]{A_n}$, $\equiv \sqrt[p]{B_1}$, $\sqrt[q]{B_2}$, $\cdots$ $\sqrt[v]{B_n}$, be any simple non-conformable surds; and let C, as well as a_1, A_1, B_1, $\cdots$ a_n, A_n, B_n, be rational:

then the relation $\sqrt[p]{B_1}+\sqrt[q]{B_2}+\cdots+\sqrt[v]{B_n}=C$ is impossible.

(*a*) *One surd*, $C \neq 0$; *or two surds*, $C=0$;

i.e., $\sqrt[p]{B_1}=C$ is impossible, Q.E.D. [df. surd

and $\sqrt[p]{B_1}+\sqrt[q]{B_2}=0$ is impossible. Q.E.D. [th.17 cr.1

(*b*) *Two quadratic surds.*

If possible, let $\sqrt{B_1}+\sqrt{B_2}=C$,

then $B_1+2\sqrt{B_1B_2}+B_2=C^2$,

$\therefore$ $2\sqrt{B_1B_2}=C^2-B_1-B_2$,

i.e., a surd equals a rational number, [th.17 cr.2

which is impossible; [df. surd

$\therefore$ $\sqrt{B_1}+\sqrt{B_2}\neq C$. Q.E.D.

(*c*) *Three quadratic surds.*

If possible, let $\sqrt{B_1}+\sqrt{B_2}+\sqrt{B_3}=C$;

then $\because$ $\sqrt{B_2}+\sqrt{B_3}=C-\sqrt{B_1}$, [hyp.

$\therefore$ $B_2+2\sqrt{B_2B_3}+B_3=C^2-2C\sqrt{B_1}+B_1$,

$\therefore$ $2C\sqrt{B_1}+2\sqrt{B_2B_3}=C^2+B_1-B_2-B_3$.

So, $2C\sqrt{B_2}+2\sqrt{B_3B_1}=C^2+B_2-B_3-B_1$;

$2C\sqrt{B_3}+2\sqrt{B_1B_2}=C^2+B_3-B_1-B_2$;

i.e., the sum of two non-conformable surds is rational,

which is impossible; [(*b*)

or else $\sqrt{B_1}$ is conformable to $\sqrt{B_2B_3}$, $\sqrt{B_2}$ to $\sqrt{B_3B_1}$, $\sqrt{B_3}$ to $\sqrt{B_1B_2}$,

and $C^2+B_1-B_2-B_3=C^2+B_2-B_3-B_1=C^2+B_3-B_1-B_2=0$,

whence $B_1=B_2=B_3$,

and $\sqrt{B_1}\pm\sqrt{B_1}\pm\sqrt{B_1}=C$,

which is also impossible; [(*a*)

$\therefore$ $\sqrt{B_1}+\sqrt{B_2}+\sqrt{B_3}\neq C$. Q.E.D.

(*d*) *Any number* n *of quadratic surds*, $c = 0$.

1. *The assumed surd equations*

$$\sqrt{B_1}+\sqrt{B_2}+\sqrt{B_3}=0,\quad \sqrt{B_1}+\sqrt{B_2}+\sqrt{B_3}+\sqrt{B_4}=0,\cdots,$$
$$\sqrt{B_1}+\sqrt{B_2}+\sqrt{B_3}+\cdots+\sqrt{B_n}=0$$

may be reduced to the equivalent surd equations

$$R_3 = S_3\sqrt{B_2 B_3},\quad R_4 = S_4\sqrt{B_3 B_4},\quad \cdots,\quad R_n = S_n\sqrt{B_{n-1}B_n},$$

and to the rational equations

$$T_3 = 0,\quad T_4 = 0,\quad \cdots,\quad T_n = 0,$$

wherein R_3, S_3, T_3 *are rational functions of* B_1, B_2, B_3;

$R_4, S_4, T_4,$ *of* $B_1, \cdots, B_4$; $\cdots$; $R_n, S_n, T_n,$ *of* $B_1, \cdots B_n$.

For if $\sqrt{B_1}+\sqrt{B_2}+\sqrt{B_3}=0$,

then $\because$ $\sqrt{B_1} = -\sqrt{B_2}-\sqrt{B_3}$,

$\therefore$ $B_1 - B_2 - B_3 = 2\sqrt{B_2 B_3}$,

i.e., $R_3 = S_3\sqrt{B_2 B_3}$, Q.E.D.

and $(B_1 - B_2 - B_3)^2 - 4B_2B_3 = 0$,

i.e., $T_3 = 0$. Q.E.D.

So, in the last two equations replace $\sqrt{B_3}$ by $\sqrt{B_3}+\sqrt{B_4}$;

then $\because$ $[B_1 - B_2 - (\sqrt{B_3}+\sqrt{B_4})^2]^2 - 4B_2(\sqrt{B_3}+\sqrt{B_4})^2 = 0$,

$\therefore$ $\Sigma B_1^2 - 2\Sigma B_1B_2 + 8B_3B_4 = 4(B_1 + B_2 - B_3 - B_4)\sqrt{B_3 B_4}$,

i.e., $R_4 = S_4\sqrt{B_3 B_4}$, Q.E.D.

and $R_4^2 - S_4^2 B_3 B_4 = 0$,

i.e., $T_4 = 0$. Q.E.D.

So, if the law holds true for k surds, it holds true for $k+1$ surds.

For in the equation $T_k = 0$ replace $\sqrt{B_k}$ by $\sqrt{B_k}+\sqrt{B_{k+1}}$,

i.e., replace B_k by $B_k + B_{k+1} + 2\sqrt{B_k B_{k+1}}$;

then $\because$ $T_k = 0$ becomes $R_{k+1} = S_{k+1}\sqrt{B_k B_{k+1}}$, Q.E.D.

$\therefore$ $R^2_{k+1} - S^2_{k+1}B_kB_{k+1} = 0$,

i.e., $T_{k+1} = 0$. Q.E.D.

But $\because$ the law holds true for 3 surds and for 4 surds,

$\therefore$ it holds true for 5 surds, for 6 surds, ... for n surds.

Q.E.D.

2. *The assumed surd equations*

$$R_3 = S_3\sqrt{B_2B_3},\quad R_4 = S_4\sqrt{B_3B_4},\quad \cdots\ R_n = S_n\sqrt{B_{n-1}B_n}$$

are all impossible.

For $\because$ $\sqrt{B_1}, \sqrt{B_2}, \sqrt{B_3}, \cdots \sqrt{B_n}$ are non-conformable surds,

$\therefore$ $\sqrt{B_2B_3}, \sqrt{B_3B_4}, \cdots \sqrt{B_{n-1}B_n}$ are surds; [th.17 cr. 2

$\therefore$ in each of these assumed equations a rational number stands equal to a surd, which is impossible, [df. surd

or else $R_n, R_n', R_n'', \cdots, S_n, S_n', S_n'', \cdots$, all $= 0$,

wherein $R_n', \cdots$ are what R_n, S_n become when $B_1, \cdots B_n$ are permuted;

e.g., S_4, S_4', S_4'' are $B_1 + B_2 - B_3 - B_4$, $B_1 + B_3 - B_2 - B_4$, $B_1 + B_4 - B_2 - B_3$;

and if $R_n, R_n', R_n'', \cdots, S_n, S_n', S_n'', \cdots$ all $= 0$,

then $B_1 = B_2 = \cdots = B_n$,

and $\sqrt{B_1}, \cdots \sqrt{B_n}$ are conformable, which is impossible. [hyp.

$\therefore$ $\sqrt{B_1} + \sqrt{B_2} + \sqrt{B_3} \neq 0,\ \sqrt{B_1} + \sqrt{B_2} + \sqrt{B_3} + \sqrt{B_4} \neq 0, \cdots,$

$\sqrt{B_1} + \sqrt{B_2} + \cdots + \sqrt{B_n} \neq 0.$ Q.E.D.

(*e*) *Any number of quadratic surds*, $C \neq 0$.

Take $\sqrt{D_{n+1}}$ a simple surd, and $D_1 \equiv B_1 \cdot D_{n+1} : C^2, \cdots D_n \equiv B_n \cdot D_{n+1} : C^2$;

then $\because$ $\sqrt{D_1} + \cdots + \sqrt{D_n} \neq \sqrt{D_{n+1}}$, [(*d*)

$\therefore$ $\sqrt{B_1} + \cdots + \sqrt{B_n} \neq C$. Q.E.D. [mult. by $(C : \sqrt{D_{n+1}})$

(*f*) *Any number* n, *of surds not all quadratic*, $C = 0$.

1. *The assumed surd equations*

$$\sqrt[p]{B_1} + \sqrt[q]{B_2} + \sqrt[r]{B_3} = 0,\quad \sqrt[p]{B_1} + \sqrt[q]{B_2} + \sqrt[r]{B_3} + \sqrt[s]{B_4} = 0,$$

$$\sqrt[p]{B_1} + \sqrt[q]{B_2} + \sqrt[r]{B_3} + \cdots + \sqrt[v]{B_n} = 0,$$

wherein each simple surd is in its lowest terms, may be reduced respectively to the equivalent surd equations

$$R_3 = S_3 \cdot V_3,\quad R_4 = S_4 \cdot V_4,\quad \cdots,\quad R_n = S_n \cdot V_n,$$

and to the equivalent rational equations

$$T_3 = 0,\quad T_4 = 0,\quad \cdots,\quad T_n = 0,$$

wherein R_3, S_3, T_3 *are rational functions of* B_1, B_2, B_3; $\cdots$;

R_4, S_4, T_4, *of* $B_1, \cdots, B_4$; $\cdots$, R_n, S_n, T_n, *of* $B_1, \cdots, B_n$.

and $V_3 \equiv$ *the surd* $B_2^{-\frac{1}{q}} \cdot B_3^{\frac{1}{r}}$, $= \sqrt[h]{B_2^{-r'} B_3^{q'}}$,

$V_4 \equiv$ *the surd* $B_3^{-\frac{1}{r}} \cdot B_4^{\frac{1}{s}}$, $= \sqrt[k]{B_3^{-s''} B_4^{r''}}, \cdots$,

and $h \equiv$ *l. c. mlt. of* q, r; q′, r′ $\equiv$ *the integers* h : r, h : q;

$k \equiv$ *l. c. mlt. of* r, s; r″, s″ $\equiv$ *the integers* k : s, k : r; $\cdots$.

For in the assumed surd equation $\sqrt[p]{B_1}+\sqrt[q]{B_2}=0$, and in the equivalent rational equation $B_1^q=\pm B_2^p$, replace B_2 by $(\sqrt[q]{B_2}+\sqrt[r]{B_3})^q$, $=B_2(1+V_3)^q$;

then $\quad B_1^q=\pm B_2^p(1+V_3)^{pq}$,

$\therefore\ B_1^q=\pm B_2^p[1+pq\,V_3+\frac{pq\cdot(pq-1)}{2!}V_3^2+\cdots]$. [bin. th.

But this equation can contain not more than $h-1$ surds;

for $\quad$ if $V_3^h, V_3^{h+1}, V_3^{h+2}, \cdots$ be present they are conformable to $V_3^0, V_3^1, V_3^2, \cdots$;

$\therefore$ the equation, reversed, reduces to the form

$$V_3^{h-1}+A_1V_3^{h-2}+A_2V_3^{h-3}+\cdots+A_{h-2}\cdot V_3+A_{h-1}=0,$$

wherein $\quad A_1, A_2, A_3, \cdots A_{h-1}$ are rational.

Let $X\equiv V_3^{h-1}+\cdots+A_{h+1}$; $\quad W\equiv V_3^h-B_2^{-r'}B_3^{q'}, =0$. $\quad$ [df. V_3

Divide W by X: the remainder Y has no power of V_3 above V_3^{h-2}.

So, divide X by Y: the rem'der Z has no pwr. of V_3 above V_3^{h-3}; ...;

and $\because\ \sqrt[h]{B_2^{-r'}B_3^{q'}}$ is a simple surd in its simplest form,

$\therefore\ V_3^h-B_2^{-r'}B_3^{q'}$, or W, has no rational factor, [df. sim. form

$\therefore$ W, X have no rational common factor;

$\therefore$ the divisions go on till a remainder is reached having only the first power of V_3; and then, one free from V_3.

Let $R_3-S_3V_3, T_3\equiv$ these remainders, wherein R_3, S_3, T_3 are rational;

then $\because\ W=0,\ X=0,\ \therefore$ each successive remainder is 0,

i.e., $\quad R_3-S_3V_3=0,\ T_3=0$. $\quad$ Q.E.D.

So, in the assumed surd equation $\sqrt[p]{B_1}+\sqrt[q]{B_2}+\sqrt[r]{B_3}=0$, and the equivalent rational equation $T_3=0$, replace B_3 by $(\sqrt[r]{B_3}+\sqrt[s]{B_4})^s$, $=B_3(1+V_4)^r$;

then $\quad$ the surd equation $\sqrt[p]{B_1}+\sqrt[q]{B_2}+\sqrt[r]{B_3}+\sqrt[s]{B_4}=0$ is equivalent to an equation $X'=0$ with no surds but $V_4, \cdots V_4^{k-1}$.

Let $W'\equiv V_4^k-B_3^{-s''}\cdot B_4^{r''}$; and divide W' by X', X' by Y', ...;

then $\quad$ the final remainders give $R_4-S_4V_4=0,\ T_4=0$. Q.E.D.

So for any number of surds. $\quad$ Q.E.D.

2. *The equations* $R_3=S_3V_3$, $R_4=S_4V_4$, ⋯ *are all impossible.*

For $\quad R_3, R_4, \cdots$ are rational, and $S_3V_3, S_4V_4, \cdots$ are surds.

(*g*) ***Any number of surds not all quadratic***, $c \neq 0$.

For, if possible, let $\sqrt[p]{B_1} + \sqrt[q]{B_2} + \cdots + \sqrt[v]{B_n} = c$; and multiply by $\sqrt[k]{B_{n+1}}$, any surd non-conformable to the others;

then $$\sqrt[k]{B_{n+1}} \cdot \sqrt[p]{B_1} + \cdots + \sqrt[k]{B_{n+1}} \cdot \sqrt[v]{B_n} - \sqrt[k]{B_{n+1}} \cdot c = 0,$$
wherein each term may reduce to a simple surd.

But this last equation is impossible; [(*f*)

$\therefore$ the given equation is impossible.

NOTE. From principles developed in X., XIII., it would appear that T_n, with perhaps a numerical coefficient, is the continued product of some or all of the $pqr \cdots v$ different values of the polynomial $\sqrt[p]{B_1} + \cdots + \sqrt[v]{B_n}$ got by combining each of the p values of $\sqrt[p]{B_1}$ with each of the q values of $\sqrt[q]{B_2}, \cdots$.

E.g., if $\sqrt{B_1} + \sqrt{B_2} + \sqrt{B_3} = 0$ be assumed true,

then $$T_3 \equiv (\sqrt[+]{B_1} + \sqrt[+]{B_2} + \sqrt[+]{B_3}) \cdot (\sqrt[+]{B_1} + \sqrt[+]{B_2} + \sqrt[-]{B_3})$$
$$\cdot (\sqrt[+]{B_1} + \sqrt[-]{B_2} + \sqrt[+]{B_3}) \cdot (\sqrt[+]{B_1} + \sqrt[-]{B_2} + \sqrt[-]{B_3}) = 0. \text{ [(}d\text{)}$$

COR. 1. *If two irreducible polynomial surds be equal, each simple surd in one polynomial equals a simple surd in the other polynomial; and the rational terms, if any, are equal.*

COR. 2. *A simple surd cannot be the sum of a rational number and a simple surd, nor of two simple surds.* [(*b*, *d*)

COR. 3. *If* A, B, a, b, a', b', $\cdots$ *be rational;* $\sqrt{c}$ *a quadratic surd;* m, n *integers; and* F *any rational function with no irrational coefficients, then:*

(*a*) *When* $$A + B\sqrt{c} = (a + b\sqrt{c}) \left\{ \begin{matrix} + \\ - \\ \times \\ \div \end{matrix} \right. (a' + b'\sqrt{c}),$$

then $$A - B\sqrt{c} = (a - b\sqrt{c}) \left\{ \begin{matrix} + \\ - \\ \times \\ \div \end{matrix} \right. (a' - b'\sqrt{c}).$$

(*b*) *When* $A + B\sqrt{c} = F(a + b\sqrt{c},\ a' + b'\sqrt{c}, \cdots)$,

then $$A - B\sqrt{c} = F(a - b\sqrt{c},\ a' - b'\sqrt{c}, \cdots).$$

(*c*) *When* $A + B\sqrt{c} = (a + b\sqrt{c})^{\frac{m}{n}}$,

then $$A - B\sqrt{c} = (a - b\sqrt{c})^{\frac{m}{n}}.$$

The reader may prove (*a*) by performing the indicated operations; (*b*) by means of (*a*); and (*c*) by the binomial theorem, first raising each member to the nth power.

§ 9. OPERATIONS ON RADICALS.

PROB. 1. TO REDUCE A RADICAL TO ITS SIMPLEST FORM.

Resolve the number whose root is sought into two factors, whereof one is the highest possible perfect power of the same degree as the radical, and the other is an entire number; write the root of the first-named factor as a coefficient before the indicated root of the other. [ths. 2, 9; 3, 10; 4, 11

E.g., $\sqrt[3]{48\,a^3b^4} = \sqrt[3]{(8\,a^3b^3\cdot 6\,b)} \cdots = 2\,ab\sqrt[3]{6\,b}$.

So, $\sqrt[n]{(a^n b^m - a^n c^p)} = \sqrt[n]{[a^n\cdot(b^m - c^p)]} = a\sqrt[n]{(b^m - c^p)}$.

PROB. 2. INVERSELY, TO FREE A RADICAL FROM COEFFICIENTS.

Raise the coefficient to a power whose exponent is the root-index of the radical; multiply this power by the expression under the radical sign, and put the same radical sign over the product.

E.g., $2\,ab\sqrt[3]{6\,b} = \sqrt[3]{(8\,a^3b^3\cdot 6\,b)} = \sqrt[3]{48\,a^3b^4}$.

So, $a\sqrt[n]{(b^m - c^p)} = \sqrt[n]{[a^n\cdot(b^m - c^p)]} = \sqrt[n]{(a^n b^m - a^n c^p)}$.

PROB. 3. TO REDUCE RADICALS TO THE SAME DEGREE.

Write the radicals as fractional powers; reduce the fractional exponents to equivalent fractions having a common denominator, restore the radical signs using the common denominator as the common root-index and the new numerators as exponents. [th. 1

$$\begin{aligned} E.g.,\quad ax,\ \sqrt[3]{by},\ \sqrt[5]{(b+c)} &= ax, \qquad (by)^{\frac{1}{3}}, \quad (b+c)^{\frac{1}{5}} \\ &= (ax)^{\frac{30}{30}}, \quad (by)^{\frac{10}{30}}, \quad (b+c)^{\frac{6}{30}} \\ &= \sqrt[30]{(ax)^{30}},\ \sqrt[30]{(by)^{10}},\ \sqrt[30]{(b+c)^6}. \end{aligned}$$

PROB. 4. TO ADD (OR SUBTRACT) RADICALS.

Reduce the radicals to their simplest form; add (or subtract) like radicals by prefixing the sum (or difference) of their coefficients to the common radical factor; write unlike radicals in any convenient order. [II. prs. 1, 2

$$\begin{aligned} E.g.,\quad 3\sqrt{8} + 5\sqrt{2} - 10\sqrt{32} &= 6\sqrt{2} + 5\sqrt{2} - 40\sqrt{2} \\ &= -29\sqrt{2}. \end{aligned}$$

$$\begin{aligned} \text{So,}\quad a\sqrt[3]{b} + a^2\sqrt[3]{b^4} - a^3\sqrt[3]{b^7} &= a\sqrt[3]{b} + a^2b\sqrt[3]{b} - a^3b^2\sqrt[3]{b} \\ &= (a + a^2b - a^3b^2)\sqrt[3]{b}. \end{aligned}$$

PROB. 5. TO MULTIPLY (OR DIVIDE) RADICALS.

Reduce the radicals to the same degree; to the product (or quotient) of the coefficients annex the product (or quotient) of the radicals. [ths. 2, 9; 3, 10

E.g., $3\sqrt{8}\cdot 5\sqrt{2}\cdot {}^{-}10\sqrt{32} = -3\cdot 5\cdot 10\cdot\sqrt{(8\cdot 2\cdot 32)}$
$= -150\cdot\sqrt{512} = -2400\cdot\sqrt{2}.$

So, $ab^{\frac{1}{3}}\cdot a^2 b^{\frac{4}{3}} : a^3 b^{-\frac{7}{3}} = a^{1+2-3}\cdot b^{\frac{1}{3}+\frac{4}{3}+\frac{7}{3}} = b^4.$

PROB. 6. TO GET A POWER (OR ROOT) OF A RADICAL.

Multiply the exponent of the given radical by the exponent of the power sought. [th. 4, 11

E.g., $(3\cdot 8^{\frac{1}{2}})^3 \quad = 27\cdot 8^{\frac{3}{2}} \quad = 432\cdot 2^{\frac{1}{2}} = 432\sqrt{2}.$

So, $\sqrt[3]{(3\cdot\sqrt{8})} = \sqrt[3]{(\sqrt{72})} = \sqrt[6]{72};$

$(a^3\cdot\sqrt{b^7})^5 = a^{15}\cdot\sqrt{b^{35}} = a^{15}\cdot b^{17}\cdot\sqrt{b};$

$(a^3\cdot b^{\frac{7}{2}})^{\frac{1}{5}} \quad = a^{\frac{3}{5}} b^{\frac{7}{10}}.$

PROB. 7. TO REDUCE A FRACTION WITH A SURD DENOMINATOR TO AN EQUIVALENT FRACTION WITH A RATIONAL DENOMINATOR.

(*a*) *The denominator a monomial: Multiply both terms of the fraction by some complement of the denominator.* [§ 8, df. comp.

(*b*) *The denominator a simple binomial quadratic surd: Multiply both terms of the fraction by the conjugate of the denominator.* [§ 8, df. conjg.

(*c*) *The denominator a binomial quadratic surd containing a complex radical: Multiply both terms of the fraction by a group of conjugate radicals that, taken together, are complementary to the denominator.*

E.g., $\dfrac{a}{b^{\frac{3}{4}}} = \dfrac{a\cdot b^{\frac{1}{4}}}{b}; \quad \dfrac{a}{\sqrt{b}-\sqrt{c}} = \dfrac{a\cdot(\sqrt{b}+\sqrt{c})}{b-c};$

$$\frac{a}{\sqrt{(b+\sqrt{c})}} = \frac{a\cdot\sqrt{(b+\sqrt{c})}}{b+\sqrt{c}} = \frac{a\cdot(b-\sqrt{c})\cdot\sqrt{(b+\sqrt{c})}}{b^2-c};$$

$$\frac{a}{b-\sqrt{(c+\sqrt{d})}} = \frac{a\cdot[b+\sqrt{(c+\sqrt{d})}]}{b^2-c-\sqrt{d}}$$

$$= \frac{a\cdot[b^2-c+\sqrt{d}]\cdot[b+\sqrt{(c+\sqrt{d})}]}{(b^2-c)^2-d}.$$

(*d*) *The denominator any binomial surd: Multiply the two fractional exponents of the denominator by the l.c.mlt. of their denominators, and attach the products as exponents to the two bases; divide the sum (or difference) of the powers so found by the denominator, and multiply both terms of the fraction by the quotient.*

E.g., to rationalize the fraction $\dfrac{6^{\frac{1}{2}}}{2^{\frac{2}{3}}+3^{\frac{3}{4}}}$;

then $\because$ 12 is the l. c. mlt. of 3, 4, and $12\cdot(\frac{2}{3}, \frac{3}{4}) = 8, 9$.

and $\because$ $(2^8 - 3^9) : (2^{\frac{2}{3}} + 3^{\frac{3}{4}})$

$$= 2^{\frac{22}{3}} - 2^{\frac{20}{3}}\cdot 3^{\frac{3}{4}} + 2^{\frac{18}{3}}\cdot 3^{\frac{6}{4}} - \cdots + 2^{\frac{2}{3}}\cdot 3^{\frac{30}{4}} - 3^{\frac{33}{4}},$$

$$\therefore\ \frac{6^{\frac{1}{2}}}{2^{\frac{2}{3}}+3^{\frac{3}{4}}} = 6^{\frac{1}{2}}\cdot\frac{2^{\frac{22}{3}} - 2^{\frac{20}{3}}\cdot 3^{\frac{3}{4}} + \cdots + 2^{\frac{2}{3}}\cdot 3^{\frac{20}{4}} - 3^{\frac{33}{4}}}{2^8 - 3^9}.$$

PROB. 8. TO FIND A SQUARE ROOT OF A BINOMIAL QUADRATIC SURD.

Let $a + \sqrt{b}$ be a binomial surd, and $x + \sqrt{y} \equiv \sqrt{(a + \sqrt{b})}$, wherein x, $\sqrt{y}$ are to be found.

Square both members of this equation;

then $\because$ $x^2 + y + 2x\sqrt{y} = a + \sqrt{b}$, (1)

$\therefore$ $x^2 + y = a$, $2x\sqrt{y} = \sqrt{b}$. [th.18 cr.1

Subtract the second of these equations from the first;

then $x^2 + y - 2x\sqrt{y} = a - \sqrt{b}$. (2)

Add equations (1, 2) together, and multiply them together;

then $x^2 + y = a$, and $x^2 - y = \sqrt{(a^2 - b)}$.

Add these two equations, divide by 2, and take the square root;

then $x = \sqrt{\dfrac{a + \sqrt{(a^2 - b)}}{2}}$.

So, $\sqrt{y} = \sqrt{\dfrac{a - \sqrt{(a^2 - b)}}{2}}$,

$$\therefore\ x + \sqrt{y},\ = \sqrt{(a + \sqrt{b})},$$
$$= \sqrt{\frac{a + \sqrt{(a^2 - b)}}{2}} + \sqrt{\frac{a - \sqrt{(a^2 - b)}}{2}},$$

and $x - \sqrt{y},\ = \sqrt{(a - \sqrt{b})},$

$$= \sqrt{\frac{a + \sqrt{(a^2 - b)}}{2}} - \sqrt{\frac{a - \sqrt{(a^2 - b)}}{2}}.$$ Q.E.F.

NOTE. Sometimes a square root of a surd of the form $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ may be found.

Write $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{(a+\sqrt{b}+\sqrt{c}+\sqrt{d})}$,

then $x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}=a+\sqrt{b}+\sqrt{c}+\sqrt{d}$.

Write $x+y+z=a$, $2\sqrt{xy}=\sqrt{b}$, $2\sqrt{xz}=\sqrt{c}$, $2\sqrt{yz}=\sqrt{d}$,

and find values for x, y, z that satisfy these equations.

E.g., to find a square root of $9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15}$.

Write $x+y+z=9$, $2\sqrt{xy}=2\sqrt{3}$, $2\sqrt{xz}=2\sqrt{5}$, $2\sqrt{yz}=2\sqrt{15}$.

then $x=1$, $y=3$, $z=5$, and the root sought is $1+\sqrt{3}+\sqrt{5}$.

PROB. 9. TO FIND A CUBE ROOT OF A BINOMIAL SURD.

Let $a+\sqrt{b}$ be a binomial surd, and $x+\sqrt{y}\equiv\sqrt[3]{a+\sqrt{b}}$, wherein x, $\sqrt{y}$ are to be found.

Cube both members of this equation;

then $\because\ x^3+3xy+(3x^2+y)\sqrt{y}=a+\sqrt{b}$,

$\therefore\ x^3+3xy=a,\ (3x^2+y)\sqrt{y}=\sqrt{b}$; [th. 18, cr. 1

and $x^3+3xy-(3x^2+y)\sqrt{y}=a-\sqrt{b}$,

i.e., $(x-\sqrt{y})^3=a-\sqrt{b}$,

$\therefore\ x-\sqrt{y}=\sqrt[3]{a-\sqrt{b}}$.

But $x+\sqrt{y}=\sqrt[3]{a+\sqrt{b}}$. [hyp.

Multiply these last two equations together;

then $\because\ x^2-y=\sqrt[3]{a^2-b}=m$, say,

$\therefore\ y=x^2-m$.

Replace y by x^2-m in the equation $x^2+3xy=a$;

then $x^3+3x(x^2-m)=a$,

i.e., $4x^3-3mx=a$.

From this point on there is no general solution, but particular examples may be solved by finding a value of x by inspection from the equation $4x^3-3mx=a$.

E.g., to find the cube root of $10+6\sqrt{3}$;

then $a=10$, $b=108$, $m=\sqrt[3]{(100-108)}=-2$;

$\therefore\ 4x^3+6x=10$,

$\therefore\ x=1$, $y=3$, and $1+\sqrt{3}$ is the root sought.

§ 10. EXAMPLES.

§ 1.

1. Replace the radical signs by fractional exponents in:

$$\sqrt{a^3};\ \sqrt[3]{x^2};\ \sqrt[3]{x^3};\ \sqrt[4]{(b^3x^2y^2)};\ \sqrt[4]{(b^4+x^3+y^2)}.$$

2. Replace the fractional exponents by radical signs in:

$$a^{\frac{4}{3}};\ x^{-\frac{1}{2}};\ a^{-\frac{m}{n}};\ (a^2+b^2)^{\frac{3}{4}};\ a^{\frac{3}{4}}+b^{\frac{3}{4}};\ (a^pb^q-c^r)^{\frac{p}{q}}.$$

§ 2, THEORS. 2, 3, 4.

... 11. Multiply or divide as indicated:

3. $(21a)^{\frac{1}{2}}:(2b)^{\frac{1}{2}};\ (x+3)^{\frac{1}{2}}\cdot(x-3)^{\frac{1}{2}};\ (20ab)^{\frac{1}{2}}\cdot(5ac)^{\frac{1}{2}}.$

4. $(2abc)^{\frac{1}{2}}\cdot(3acd)^{\frac{1}{2}}\cdot(6bd)^{\frac{1}{2}};\ 2(abc)^{\frac{1}{3}}\cdot-3(a^2c)^{\frac{1}{3}}:-4(b^2c)^{\frac{1}{3}}.$

5. $(a+b)^{\frac{1}{m}}\cdot(a+b)^{\frac{1}{n}}\cdot(a-b)^{\frac{1}{m}}\cdot(a-b)^{\frac{1}{n}}\cdot(a^2+b^2)^{\frac{1}{m}+\frac{1}{n}}.$

6. $x^{\frac{3}{2}}:x^{-\frac{1}{2}};\ a^{\frac{1}{2}}\cdot a^{-\frac{1}{3}}\cdot a^{-\frac{1}{4}}:a^{-\frac{1}{5}};\ a^{\frac{1}{3}}\cdot a^{-\frac{3}{4}}:(a^{\frac{1}{12}}\cdot a^{\frac{3}{4}}\cdot a^{-\frac{49}{24}}).$

7. $(5a^{-\frac{1}{2}}+6b^{\frac{1}{3}})\cdot(5a^{-\frac{1}{2}}-6b^{\frac{1}{3}});\ (x^{\frac{1}{4}}y+y^{\frac{2}{3}}z^{\frac{1}{2}})\cdot(x^{\frac{1}{4}}-y^{-\frac{1}{3}}z^{\frac{1}{2}}).$

8. $(a^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}})\cdot(a^{\frac{1}{3}}+b^{\frac{1}{3}});\ (x^{-1}-y^{-1}):(x^{-\frac{1}{3}}-y^{-\frac{1}{3}});$
$(-3a^{-5}+2a^{-4}b^{-1})\cdot(-2a^{-3}-3a^{-4}b).$

9. $(a^{\frac{2}{3}}-a^{\frac{1}{3}}+1-a^{-\frac{1}{3}}+a^{-\frac{2}{3}})\cdot(a^{\frac{1}{3}}+1+a^{-\frac{1}{3}})\cdot(a^{\frac{1}{3}}-a^{-\frac{1}{3}})\cdot a^{\frac{2}{3}}.$

10. $(x-y):(x^{-\frac{1}{4}}-y^{-\frac{1}{4}});\ (x^9-y^{-2}):(x^{\frac{3}{2}}-y^{-\frac{1}{3}}).$

11. $(2x^5y^{-3}-5x^4y^{-2}+7x^3y^{-1}-5x^2+2xy):(x^2y^{-3}-x^2y^{-2}+xy^{-1}).$

12. Simplify the fraction:

$$\frac{a^{\frac{3}{2}}-ax^{-\frac{1}{2}}+a^{\frac{1}{2}}x^{-1}-x^{-\frac{3}{2}}}{a^{\frac{5}{2}}-a^2x^{-\frac{1}{2}}+a^{\frac{3}{2}}x^{-1}-ax^{-\frac{3}{2}}+a^{\frac{1}{2}}x^{-2}-x^{-\frac{5}{2}}}.$$

13. Get the square of:

$$(a^5-b^5)^{\frac{1}{2}};\ 3a^{\frac{2}{3}}b^{-1}x^{\frac{1}{4}};\ x-(ay)^{\frac{1}{2}};\ \tfrac{1}{2}a^{\frac{1}{2}}x^{-\frac{1}{2}}-2a^{-\frac{1}{2}}x^{\frac{1}{2}}.$$

14. Get the cube of:

$$2(3a)^{\frac{1}{3}};\ 3a^{\frac{1}{6}}b^{-\frac{1}{2}}xy^{-1};\ (a^{\frac{1}{3}}-b^{\frac{1}{3}})^{\frac{2}{3}};\ x+\sqrt{y};\ a^{-\frac{1}{2}}-x^{\frac{3}{2}}.$$

15. Express in simplest form:

$$\left(\frac{27a}{8b^3}\right)^{-\frac{1}{3}};\ \left[\left(\frac{a^{-2m}}{b^{-2n}}\right)^{-\frac{p}{m}}\right]^{\frac{q}{2n}};\ \left[\frac{2}{3}\left(\frac{3}{4}\cdot\frac{x^2}{y}\right)^{\frac{1}{3}}\right]^4;\ \frac{(x^3y^{mn})^{\frac{1}{m}}}{(x^2y^{mn})^{\frac{1}{n}}}.$$

16. Get the product of:

$$\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{(a^2b)^{\frac{3}{4}}}\cdot\frac{(a^2xy^{-1})^{-\frac{1}{2}}}{by^{\frac{1}{2}}}\cdot\left(\frac{a^{\frac{1}{6}}b^{-\frac{1}{2}}}{x^{-1}y^{\frac{1}{3}}}\right)^{-3}\cdot\left(\frac{a^4b^{\frac{1}{2}}x^5}{a^{-3}b^2x^{-4}}\right)^{\frac{1}{2}}\cdot\left(\frac{a^{-2}b^{\frac{1}{3}}x^4y^{-\frac{5}{2}}}{a^4b^{-3}x^{-\frac{1}{2}}y^2}\right)^{-\frac{1}{3}}.$$

$$[(a^{-\frac{1}{2}})^{\frac{1}{3}}+\{(a^{\frac{1}{2}}b)^{\frac{1}{2}}\}^{\frac{1}{3}}]\cdot[(a^{-\frac{1}{2}})^{\frac{1}{3}}-\{(a^{\frac{1}{2}}b)^{\frac{1}{2}}\}^{\frac{1}{3}}]\cdot(a^{\frac{1}{3}}+a^{\frac{1}{6}}b^{\frac{1}{3}}).$$

17. Get the square of:

$$2a^{\frac{1}{2}}x+3ax^{\frac{1}{2}}-2a^2x^{-\frac{1}{2}}-3a^{\frac{5}{2}}x^{-1};\quad a^{-\frac{1}{2}}b+a^{-1}b^2a^{-\frac{3}{2}}b^3.$$

$$\frac{a}{x}-2\left(\frac{a}{x}\right)^{\frac{1}{2}}+3-2\left(\frac{a}{x}\right)^{-\frac{1}{2}}+\left(\frac{a}{x}\right)^{-1};\quad \left(\frac{b^2}{a}\right)^{\frac{1}{2}}+\left(\frac{b^4}{a^2}\right)^{\frac{1}{2}}+\left(\frac{b^6}{a^3}\right)^{\frac{1}{2}}.$$

18. Get the cube of:

$$\tfrac{2}{3}x^{\frac{5}{6}}y^{-\frac{1}{6}}-\tfrac{3}{2}x^{-\frac{1}{6}}y^{\frac{5}{6}};\quad \tfrac{1}{4}a^{-\frac{3}{2}}-4x^{-1}a^{\frac{3}{2}};\quad \tfrac{1}{5}a^{-\frac{1}{2}}b^{-\frac{2}{3}}+5a^{\frac{2}{3}}b^{\frac{1}{2}}.$$

19. Find the square root of:

$$x^{-2}-6x^{-1}y^{-1}+9y^{-2};\quad a^{-6}-4a^{-5}-2a^{-4}+12a^{-3}+9a^{-2};$$

$$4a^{-1}-12a^{-\frac{1}{2}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}+16a^{-\frac{1}{2}}c^{\frac{1}{4}}-24b^{\frac{1}{3}}c^{\frac{1}{3}}+16c^{\frac{1}{2}};$$

$$a^{-\frac{2}{xy}}+(a^{\frac{1}{y}}b)^{\frac{1}{r}}-2b^{\frac{1}{2r}}x^{\frac{x-2r}{2rxy}};\quad \frac{y^2}{x}+\frac{x^2}{4y}+\frac{2y^{\frac{3}{2}}-x^{\frac{3}{2}}}{(xy)^{\frac{1}{4}}}.$$

20. Find the cube root of:

$$\tfrac{1}{8}a^3-\tfrac{3}{2}a^2b^{\frac{1}{2}}+6ab-8b^{\frac{3}{2}};\quad \tfrac{1}{27}x+\tfrac{1}{15}x^{\frac{2}{3}}y^{\frac{1}{2}}+\tfrac{1}{25}x^{\frac{1}{3}}y^1+\tfrac{1}{125}y^{\frac{3}{2}}.$$

$$\tfrac{1}{27}a^{-\frac{3}{2}}-27x^{-3}a^{\frac{3}{2}}-(x^2a)^{-\frac{1}{2}}+(3x^{-1}a^{\frac{1}{4}})^2.$$

21. Find the fourth root of:

$$\tfrac{1}{16}x^3-\tfrac{5}{2}x^{\frac{9}{4}}y^{-\frac{4}{5}}+\tfrac{75}{2}x^{\frac{3}{2}}y^{-\frac{8}{5}}-250x^{\frac{3}{4}}y^{-\frac{12}{5}}+625y^{-\frac{16}{5}}.$$

§ 5, THEORS. 9, 10, 11.

...25. Multiply or divide as indicated:

22. $x^{\sqrt{2}}\cdot y^{\sqrt{2}}:z^{\sqrt{2}};\quad (24a)^{\sqrt[3]{3}}:(6b)^{\sqrt[3]{3}};\quad (x+3)^{\pi}\cdot(x-3)^{\pi}.$

23. $x^{2\sqrt{3}}\cdot x^{3\sqrt{3}}:x^{4\sqrt{3}};\quad a^{\frac{1}{\sqrt{2}}}\cdot a^{\frac{1}{\sqrt{3}}}:a^{\frac{1}{2}};\quad (a^{\sqrt{2}}-b^{\sqrt{3}})\cdot(a^{\sqrt{2}}+b^{\sqrt{3}}).$

24. $10^{1.30103\ldots}\cdot 10^{2.69897\ldots},=20\cdot 500;\quad 10^{2.69897\ldots}:10^{1.30103\ldots},=500:20.$

25. $(x^{\sqrt{2}}-y^{\sqrt{2}}):(x^{\sqrt{\frac{1}{2}}}-y^{\sqrt{\frac{1}{2}}});\quad [a^{\sqrt{2}}+2(ab)^{\sqrt{\frac{1}{2}}}+b^{\sqrt{2}}]:(a^{\sqrt{\frac{1}{2}}}+b^{\sqrt{\frac{1}{2}}}).$

...28. Get the powers and roots as indicated:

26. $(10^{1.30103\ldots})^2,=20^2;\quad (10^{.30103})^{\frac{1}{2}},=\sqrt{2};\quad (10^{\sqrt{2}})^{2\sqrt{2}};\quad (10^{\pi})^{\pi}.$

27. $(2a^{\sqrt{\frac{1}{2}}}x+3ax^{\sqrt{\frac{1}{2}}})^2;\quad (4a^{\sqrt{2}}+12a^{\sqrt{\frac{1}{2}}}x^{\sqrt{\frac{1}{2}}}+9x^{\sqrt{2}})^{\frac{1}{2}};\quad (x^{\sqrt{\frac{1}{2}}}y^{\sqrt{2}})^{\sqrt{\frac{1}{2}}}.$

28. $(\tfrac{2}{3}x^{\sqrt[3]{\frac{1}{3}}}+\tfrac{3}{2}y^{\sqrt[3]{\frac{2}{3}}})^3;\quad (a^{\sqrt{3}}-3a^{2\sqrt{\frac{1}{3}}}b^{\sqrt{\frac{1}{3}}}+3a^{\sqrt{\frac{1}{3}}}b^{2\sqrt{\frac{1}{3}}}-b^{\sqrt{3}})^{\frac{1}{3}}.$

§ 7, THEORS. 14, 15, 16.

In the following examples e is assumed to be such a number that $\mathrm{M}_e = 1$, and a is any constant.

...32. Find the derivatives as to x of the variable powers:

29. e^x; a^x; x^x; e^{-x}; a^{-x}; x^{-x}; $e^{\frac{1}{x}}$; $a^{\frac{1}{x}}$; $x^{\frac{1}{x}}$; $e^{-\frac{1}{x}}$; $a^{-\frac{1}{x}}$; $x^{-\frac{1}{x}}$.

30. e^{e^x}; e^{a} ; a^{e^x}; a^{a^x}; e^{x^x}; a^{x^x}; x^{e^x}; x^{a^x}; x^{x^x}.

31. $e^x(1-a^x)$; $e^{(a+x)^2}$; e^{-a^2x}; $a^{\sqrt{(a^2-x^2)}}$; $a^{-\sqrt{(a^2-x^2)}}$.

32. $(ax)^{ax}$; $(ax)^{-ax}$; $\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$; $\dfrac{a^x-a^{-x}}{a^x+a^{-x}}$; $\left(\dfrac{a}{x}\right)^{\frac{a}{x}}$; $\left(\dfrac{x}{a}\right)^{\frac{x}{a}}$; $\left(\dfrac{a}{x}\right)^{\frac{x}{a}}$.

...35. Find the derivatives as to x of the logarithms:

33. $\log_e e^x$; $\log_a a^x$; $\log_{\mathrm{A}} x^x$; $\log_e e^{-x}$; $\log_a a^{-x}$; $\log_{\mathrm{A}} x^{-x}$.

34. $\log_e(a+bx^n)$; $\log_e[\log_e(a+bx^n)]$; $\log_e(e^x-e^{-x})$.

35. $\log_e\dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$; $\log_e\dfrac{1+x\sqrt{2+x^2}}{1-x\sqrt{2+x^2}}$; $\log_e\dfrac{x^x-x^{-x}}{x^x+x^{-x}}$.

§ 9, PROB. 1.

...39. Reduce to simplest form:

36. $125^{\frac{1}{2}}$; $567^{\frac{1}{2}}$; $392^{\frac{1}{2}}$; $1008^{\frac{1}{2}}$; $216^{\frac{1}{3}}$; $72^{\frac{1}{3}}$; $162^{\frac{1}{3}}$; $48^{\frac{1}{4}}$; $160^{\frac{1}{5}}$.

37. $(11\tfrac{13}{16})^{\frac{1}{3}}$ $(6\tfrac{21}{64})^{\frac{1}{3}}$; $(10\tfrac{1}{8})^{\frac{3}{4}}$; $(6\tfrac{62}{75})^{-\frac{2}{3}}$; $2500^{\frac{1}{4}}$; $\sqrt[3]{296352}$.

38. $\sqrt{147x^{-3}yz^2}$; $\sqrt[3]{56a^4b^5c^6}$; $\sqrt[4]{112a^{-5}b^{-2}c^3}$; $\sqrt[6]{64a^8b^{-3}c^4}$.

39. $\sqrt{(72a^2b-72b+18a^{-2}b)}$; $\sqrt[3]{[x^4y^{-1}-xy^2-3x^2(x-y)]}$.

§ 9, PROB. 2.

...43. Free from coefficients:

40. $6\sqrt{5}$; $2\sqrt{x}$; $2x\sqrt{2}$; $4a\sqrt{5b}$; $4\sqrt[3]{6}$; $5a\sqrt[3]{y}$; $\frac{3}{2}\sqrt{9\frac{1}{3}}$.

41. $\frac{1}{2}\sqrt{2b}$; $5\sqrt{\frac{1}{5}c}$; $27a^{\frac{3}{4}}$; $\frac{1}{10}(\frac{5}{9})^{\frac{1}{3}}$; $\frac{9}{5}(\frac{5}{6})^{\frac{1}{2}}$; $\frac{2}{3}xy^{-1}\cdot(\frac{3}{2}yx^{-1})^{\frac{2}{3}}$.

42. $3a^2\sqrt[3]{2a^2d^2}$; $\frac{1}{2}\sqrt[3]{4x^2y}$; $5a\sqrt[3]{\frac{1}{10}ay}$; $\frac{1}{3}abc\sqrt[4]{3a^2b}$.

43. $\dfrac{a-b}{a+b}\sqrt{\dfrac{a+b}{a-b}}$; $\dfrac{x^3y^2}{z^2}\cdot\left(\dfrac{z^5}{x^5y^5}\right)^{\frac{1}{2}}$; $(x^2-y^2)^{\frac{3}{2}}\cdot\sqrt{\dfrac{x-y}{x^2+2xy+y^2}}$.

§ 9, PROB. 3.

...45. Reduce to the same degree:

44. $a^{\frac{1}{2}}, a^{\frac{1}{3}}$; $a^{\frac{1}{3}}, b^{\frac{1}{4}}$; $3^{\frac{1}{2}}, 4^{\frac{1}{4}}$; $\sqrt[3]{ab}$, $\sqrt[4]{ac}$, $\sqrt[5]{bc}$, $\sqrt[6]{(b+c)}$.

45. $a^{\frac{1}{2}}, b^{\frac{2}{3}}$; $a^{\frac{5}{6}}, b^{\frac{3}{4}}, c^{\frac{5}{3}}$; $x^{\frac{1}{2}}, x^{\frac{2}{3}}, x^{\frac{3}{4}}, x^{\frac{4}{5}}, x^{\frac{5}{6}}$; $(3x)^{\frac{1}{2}}, 2y^{\frac{2}{3}}, 4z^{\frac{3}{4}}$.

46. Which is the greater: $(\frac{1}{2})^{\frac{1}{2}}$ or $(\frac{2}{3})^{\frac{2}{3}}$? $\sqrt[5]{2}$ or $\sqrt[8]{3}$?
$\sqrt[3]{9}$ or $\sqrt[4]{18}$? $m^{\frac{1}{2}}$ or $(m+1)^{\frac{1}{3}}$ when $m > 3$?

§ 9, PROB. 4.

...51. Add or subtract as indicated:

47. $\sqrt{18}-\sqrt{8}$; $\sqrt{128}-2\sqrt{50}+7\sqrt{72}$; $6\sqrt{\frac{3}{4}}-3\sqrt{\frac{4}{3}}$.

48. $9\sqrt{80}-2\sqrt{125}-5\sqrt{245}+\sqrt{320}$; $3\sqrt{\frac{2}{5}}+4\sqrt{\frac{1}{10}}$.

49. $625^{\frac{1}{3}}-7\cdot 135^{\frac{1}{3}}+8\cdot 320^{\frac{1}{3}}$; $8\cdot(\frac{3}{4})^{\frac{1}{2}}+\frac{1}{2}\cdot 12^{\frac{1}{2}}-\frac{4}{3}\cdot 27^{\frac{1}{2}}$.

50. $b(8a^6b)^{\frac{1}{3}}+4a(a^3b^4)^{\frac{1}{3}}-125(a^6b^4)^{\frac{1}{3}}$; $a^2b^{\frac{1}{3}}+2ab^{\frac{4}{3}}+b^{\frac{7}{3}}$.

51. $2\sqrt[3]{\frac{1}{5}}+3\sqrt[3]{\frac{1}{40}}$; $\frac{9}{5}(\frac{500}{243})^{\frac{1}{3}}-\frac{11}{9}(\frac{108}{125})^{\frac{1}{3}}$; $\frac{3}{8}ab^{\frac{1}{3}}-\frac{2}{9}(b:a^{-3})^{\frac{1}{3}}$.

§ 9, PROB. 5.

... 59. Multiply or divide as indicated:

52. $3\sqrt{2}\cdot 2\sqrt{3}$; $8\sqrt{6}:2\sqrt{2}$; $5\sqrt{7}\cdot 2\sqrt{7}$; $3^{\frac{1}{3}}\cdot 2^{\frac{1}{2}}$.

53. $3\sqrt{6}\cdot 2\sqrt{3}\cdot 4\sqrt{5}:12\sqrt{10}$; $4\sqrt{3}\cdot 3\sqrt{5}\cdot 5\sqrt[3]{2}$; $2^{\frac{1}{2}}\cdot 3^{\frac{1}{3}}\cdot 4^{\frac{1}{4}}$.

54. $\frac{1}{10}\cdot(\frac{5}{9})^{\frac{1}{3}}:(\frac{9}{5})\cdot(\frac{15}{18})^{\frac{1}{2}}$; $5^{\frac{3}{4}}\cdot 4^{\frac{2}{3}}\cdot 3^{\frac{3}{2}}:60^{\frac{5}{6}}$; $(\frac{3}{2})^{\frac{3}{4}}:(\frac{3}{4})^{\frac{2}{3}}$.

55. $(\frac{5}{14})\cdot(\frac{2}{3})^{\frac{2}{3}}:\frac{5}{21}(\frac{9}{4})^{-\frac{1}{3}}$; $\sqrt{(a^2-b^2)}:\sqrt{(a-b)}:\sqrt[4]{(a-b)}$.

56. $\frac{2}{3}a\sqrt[3]{b^2}\cdot\frac{3}{4}b\sqrt{a^3}\cdot\frac{2}{5}a^{-\frac{1}{2}}b^{-\frac{1}{3}}$; $(5+2\sqrt{2})\cdot(5-2\sqrt{2})$.

57. $(2+\sqrt{3})^2$; $(8\sqrt{2}+2\sqrt{3})\cdot(2\sqrt{2}+\sqrt{3})$.

58. $(4+\sqrt{2})\cdot(1-\sqrt{3})\cdot(4-\sqrt{2})\cdot(5-\sqrt{3})\cdot(1+\sqrt{3})(5+\sqrt{3})$.

59. $(a+b)^{\frac{1}{m}}\cdot(a+b)^{\frac{1}{n}}\cdot(a-b)^{\frac{1}{m}}\cdot(a-b)^{\frac{1}{n}}\cdot(a^2+b^2)^{\frac{1}{mn}}$;
$\sqrt{-a}\cdot\sqrt{-b}\cdot\sqrt[4]{-a}\cdot\sqrt[4]{-b}\cdot\sqrt[6]{-a}\cdot\sqrt[6]{-b}\cdot\sqrt[8]{-a}\cdot\sqrt[8]{-b}$.

§ 9, PROB. 6.

...64. Find the required powers or roots:

60. $(3\sqrt{3})^4$; $(2\sqrt[3]{5})^6$; $(\sqrt{2}-\sqrt{3})^2$; $(\sqrt{10}-\sqrt{5})^2$; $(3^{\frac{1}{2}}-3^{-\frac{1}{2}})^2$.

61. $(\sqrt{\frac{4}{3}}-\sqrt{\frac{3}{4}})^2$; $(2^{\frac{1}{3}}-2^{-\frac{2}{3}})^3$; $(3^{\frac{1}{3}}-3^{-\frac{1}{3}})^3$; $(4^{\frac{1}{3}}+4^{-\frac{1}{3}})^4$.

62. $(\sqrt{\frac{2}{3}}x-\frac{3}{2}y)^2$; $[a^3b(a^3bc)^{\frac{1}{5}}]^{\frac{1}{6}}$; $(2x^{\frac{1}{2}}y^{\frac{1}{3}}z^{\frac{1}{4}})^4$; $[(5x^{\frac{1}{2}}y^{-\frac{1}{4}})^{\frac{1}{2}}]^{-\frac{1}{3}}$.

63. $\sqrt[m]{-2^{\frac{1}{m}}a^mb^mc^{2m}}$; $\sqrt[3]{(27a^3x)^{\frac{1}{2}}}$; $(a^{\frac{1}{3}}x^{-1}+a^{-\frac{1}{3}}x)^3$.

64. $[(a+b)^{\frac{1}{2}}-(a-b)^{\frac{1}{2}}]^2$; $(a^{\frac{1}{3}}+b^{\frac{1}{3}}-c^{\frac{1}{3}})^3$; $(a^{\frac{1}{3}}b^{-\frac{1}{3}}+a^{-\frac{1}{3}}b^{\frac{1}{3}})^3$.

§ 9, PROB. 7.

··· 69. Reduce to equivalent fractions with rational denominators :

65. $\dfrac{1}{\sqrt{3}}$; $\dfrac{6}{\sqrt{2}}$; $\dfrac{3}{2\sqrt{3}}$; $\dfrac{3\sqrt{8}}{2\sqrt{2}}$; $\dfrac{2x}{3y^{\frac{1}{2}}}$; $\dfrac{4}{5^{\frac{2}{3}}}$; $\dfrac{4x}{3y^{\frac{3}{4}}}$; $\left(\dfrac{a}{y}\right)^{\frac{2}{3}}$; $\left(\dfrac{m}{n}\right)^{-\frac{4}{5}}$.

66. $\dfrac{2}{\sqrt{3}+1}$; $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$; $\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$; $\dfrac{21}{\sqrt{10}-\sqrt{3}}$; $\dfrac{1+\sqrt{5}}{3\sqrt{5}-2\sqrt{3}}$.

67. $\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$; $\dfrac{(a+b)^{\frac{1}{2}}+(a-b)^{\frac{1}{2}}}{(a+b)^{\frac{1}{2}}-(a-b)^{\frac{1}{2}}}$.

68. $\dfrac{(3+\sqrt{3})\cdot(3+\sqrt{5})\cdot(\sqrt{5}-2)}{(5-\sqrt{5})\cdot(1+\sqrt{3})}$; $\dfrac{1}{a+\sqrt{[b+\sqrt{(c+\sqrt{d})}]}}$.

69. $\dfrac{\sqrt{2}\cdot(\sqrt{3}+1)\cdot(2-\sqrt{3})}{(\sqrt{2}-1)\cdot(3\sqrt{3}-5)\cdot(2+\sqrt{2})}$; $\dfrac{\sqrt{2}\cdot(\sqrt{2}-3)}{(\sqrt{2}+8)\cdot(\sqrt{3}-\sqrt{5})}$.

··· 71. Reduce to simplest form :

70. $\dfrac{1}{a-(a^2-x^2)^{\frac{1}{2}}}-\dfrac{1}{a+(a^2-x^2)^{\frac{1}{2}}}$; $\dfrac{x+(x^2-1)^{\frac{1}{2}}}{x-(x^2-1)^{\frac{1}{2}}}$; $\dfrac{x-(x^2-1)^{\frac{1}{2}}}{x+(x^2-1)^{\frac{1}{2}}}$.

71. $\dfrac{(x^2+1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}}+\dfrac{(x^2+1)^{\frac{1}{2}}-(x^2-1)^{\frac{1}{2}}}{(x^2+1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}}$; $\dfrac{x-(x^2+1)^{\frac{1}{2}}}{x+(x^2+1)^{\frac{1}{2}}}$.

72. In the equation

$$(x^6-y^6):(x-y)=x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$$

put $x=a^{\frac{1}{2}}$ and $y=b^{\frac{1}{3}}$; thence find $(a^3-b^2):(a^{\frac{1}{2}}-b^{\frac{1}{3}})$, and apply this and similar results to reduce to equivalent fractions with rational denominators :

$\dfrac{1}{a^{\frac{1}{2}}-b^{\frac{1}{3}}}$; $\dfrac{10}{2-\sqrt[3]{6}}$; $\dfrac{3^{\frac{1}{4}}-2^{\frac{1}{4}}}{3^{\frac{1}{4}}+2^{\frac{1}{4}}}$; $\dfrac{\sqrt[3]{5}-\sqrt[3]{4}}{\sqrt[3]{5}+\sqrt[3]{4}}$; $\dfrac{1}{a^3-b^{\frac{1}{3}}}$; $\dfrac{2a+b^{\frac{1}{4}}}{2a-b^{\frac{1}{4}}}$.

73. Show that

$$(\sqrt{a}+\sqrt{b}+\sqrt{c}),\ (\sqrt{a}+\sqrt{b}-\sqrt{c}),\ (a+b-c-2\sqrt{ab})$$

form a complementary group; and thence reduce to equivalent fractions with rational denominators :

$\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$; $\dfrac{1+\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$; $\dfrac{1-\sqrt{a}}{1-\sqrt{a}-\sqrt{b}}$;

$\dfrac{\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$; $\dfrac{a}{\sqrt{(b+\sqrt{c})}+\sqrt{(d+\sqrt{e})}}$.

74. Find the value of:

$$\frac{x-(1+x^2)^{\frac{1}{2}}}{x+(1+x^2)^{\frac{1}{2}}}+\frac{x+(1+x^2)^{\frac{1}{2}}}{x-(1+x^2)^{\frac{1}{2}}} \text{ when } x=\frac{1}{2}\left[\left(\frac{a}{b}\right)^{\frac{1}{2}}-\left(\frac{b}{a}\right)^{\frac{1}{2}}\right];$$

$$\frac{\frac{1}{2}+x}{\frac{1}{2}+\sqrt{(\frac{1}{2}+x)}}+\frac{\frac{1}{2}-x}{\frac{1}{2}+\sqrt{(\frac{1}{2}-x)}} \text{ when } x=\tfrac{1}{2}\sqrt{3}.$$

75. Show that $y=\frac{1}{2}(e^x-e^{-x})$ if $e^x=y+\sqrt{(1+y^2)}$.

§ 9, PROB. 8.

…78. Find the square root of:

76. $7+2\sqrt{10}$; $7+4\sqrt{3}$; $2-\sqrt{3}$; $16-6\sqrt{7}$; $\sqrt{18}-\sqrt{16}$.

77. $8\sqrt{3}-6\sqrt{5}$; $75-12\sqrt{21}$; $\sqrt{27}+\sqrt{15}$; $-9+6\sqrt{3}$.

78. $ab+c^2+\sqrt{(a^2-c^2)(b^2-c^2)}$; $2[1+(1-c^2)^{\frac{1}{2}}]$;
$xy-2x(xy-x^2)^{\frac{1}{2}}$; $1-2a\sqrt{(1-a^2)}$.

79. Find the fourth root of:

$28-16\sqrt{3}$; $49+20\sqrt{6}$; $a^2+b^2+6ab-4(a^{\frac{3}{2}}b^{\frac{1}{2}}+a^{\frac{1}{2}}b^{\frac{3}{2}})$.

80. If $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{(a+2\sqrt{b}+2\sqrt{c}+2\sqrt{d})}$,
show that x, y, z must satisfy the four conditions
$x+y+z=a$, $xy=b$, $xz=c$, $yz=d$,
and hence show that the square root of
$6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$ may be found.

81. Find the square root of:

$10+2\sqrt{6}+2\sqrt{10}+2\sqrt{15}$; $8+2\sqrt{2}+2\sqrt{5}+2\sqrt{10}$;
$15-2\sqrt{15}-2\sqrt{21}+2\sqrt{35}$; $11+2\sqrt{6}+4\sqrt{3}+6\sqrt{2}$.

82. Show that the square root of $10+2\sqrt{6}+2\sqrt{14}+2\sqrt{21}$ cannot be expressed in the form $\sqrt{a}+\sqrt{b}+\sqrt{c}$.

83. Find the square root of:

$15-2\sqrt{3}-2\sqrt{15}+6\sqrt{2}-2\sqrt{6}+2\sqrt{5}-2\sqrt{30}$.

§ 9, PROB. 9.

84. Find the cube root of:

$7+5\sqrt{2}$; $16+8\sqrt{5}$; $22+10\sqrt{7}$; $38+17\sqrt{5}$;
$21\sqrt{6}-23\sqrt{5}$; $3a-2a^3+(1+2a^2)\sqrt{(1-a^2)}$.

IX. LOGARITHMS.

§ 1. GENERAL PROPERTIES.

THE LOGARITHM of a number is the exponent of that power to which another number, the *base*, must be raised to give the number first named. [I. § 11

E.g., in the equation $A^x = N$, A is the base, N the number; and x the exponent of the power of A and the *logarithm to base* A of the number N.

The equation $x = \log_A N$ expresses the relation last named.

The equation $N = \log_A^{-1} x$ means that N is the number, A^x, whose logarithm to base A is x; it is read, N is the *anti-logarithm* of x to base A.

E.g., $0 = \log_A 1$ and $A = \log^{-1} 0$, whatever A may be.

So, $1 = \log_2 2$, $2 = \log_3 9$, $3 = \log_4 64$, $4 = \log_5 625, \cdots$,
and $2 = \log_2^{-1} 1$, $9 = \log_3^{-1} 2$, $64 = \log_4^{-1} 3$, $625 = \log_5^{-1} 4$, $\cdots$.

So, $^-1 = \log_2 \frac{1}{2}$, $^-2 = \log_3 \frac{1}{9}$, $^-3 = \log_4 \frac{1}{64}$, $^-4 = \log_5 \frac{1}{625}, \cdots$,
and $^-1 = \log_{\frac{1}{2}} 2$, $^-2 = \log_{\frac{1}{3}} 9$, $^-3 = \log_{\frac{1}{4}} 64$, $^-4 = \log_{\frac{1}{5}} 625 \cdots$.

If the base be well known it may be suppressed, and these two equations may then be written $x = \log N$, $N = \log^{-1} x$.

If while A is constant N take in succession all possible values from 0 to ∞, the corresponding values of x when taken together constitute *a system of logarithms to base* A.

Operations upon or with logarithms are therefore operations upon or with the exponents of the powers of any same base; and the principles established for such powers apply directly to logarithms, with but the change of name noted above.

THEOR. 1. *The logarithm of unity to any base is zero.* [df. pwr.

THEOR. 2. *The logarithm of any number to itself as base is unity.* [df. pwr.

THEOR. 3. *To any positive base* $\left\{ \begin{matrix} \textit{larger} \\ \textit{smaller} \end{matrix} \right.$ *than unity, every positive number has one and but one real logarithm:* $\left\{ \begin{matrix} \textit{an increasing} \\ \textit{a decreasing} \end{matrix} \right.$ *function of the number.* [VIII. th. 13

NOTE. If either the base or the number be negative, there may or may not be one real logarithm.

E.g.,	$^{+}100$ has the logarithm 2 to base $^{+}10$ or $^{-}10$,
and	both $^{+}10$ and $^{-}10$ have the logarithm $\frac{1}{2}$ to base $^{+}100$;
but	$^{-}100$ has no real logarithms to base $^{+}10$ or $^{-}10$,
nor	has $^{+}10$ or $^{-}10$ a real logarithm to base $^{-}100$.

So,	$^{\pm}1000$ has the logarithm 3	to base $^{\pm}10$,
and	$^{\pm}10$ has the logarithm $\frac{1}{3}$	to base $^{\pm}1000$;
but	$^{\mp}1000$ has no real logarithm	to base $^{\pm}10$,
and	$^{\pm}10$ has no real logarithm	to base $^{\mp}1000$.

In this chapter, and in general where logarithms to the base 10 are used as aids in numerical computations, the number as well as the base is assumed to be positive unless the contrary be stated.

THEOR. 4. *If the base be positive and* $\left\{\begin{matrix}\textit{larger}\\ \textit{smaller}\end{matrix}\right.$ *than unity, the logarithms of all numbers greater than unity are* $\left\{\begin{matrix}\textit{positive};\\ \textit{negative};\end{matrix}\right.$ *of all numbers positive and less than unity,* $\left\{\begin{matrix}\textit{negative}.\\ \textit{positive}.\end{matrix}\right.$ [VIII. lem. th. 12

THEOR. 5. *If the base be positive and* $\left\{\begin{matrix}\textit{larger}\\ \textit{smaller}\end{matrix}\right.$ *than unity, and if the number be a positive variable that approaches zero, then its logarithm approaches* $\left\{\begin{matrix}\textit{negative infinity}.\\ \textit{positive infinity}.\end{matrix}\right.$ [VIII. th. 13

THEOR. 6. *The logarithm of a* $\left\{\begin{matrix}\textit{product}\\ \textit{quotient}\end{matrix}\right.$ *of two numbers is the* $\left\{\begin{matrix}\textit{sum of the logarithms of the factors}.\\ \textit{excess of log. div'd over log. div'r}.\end{matrix}\right.$ [VIII. ths. 2, 9

E.g., $\log_{A}(B \cdot C : D) = \log_{A} B + \log_{A} C - \log_{A} D$.

THEOR. 7. *The logarithm of a* $\left\{\begin{matrix}\textit{power}\\ \textit{root}\end{matrix}\right.$ *of a number is the* $\left\{\begin{matrix}\textit{product}\\ \textit{quotient}\end{matrix}\right.$ *of the logarithm of the number by the* $\left\{\begin{matrix}\textit{exponent}.\\ \textit{root-index}.\end{matrix}\right.$ [VIII. ths. 3, 10

E.g., $\log_{A}(B^2 \cdot \sqrt[3]{C}) = 2\log_{A} B + \frac{1}{3}\log_{A} C$.

COR. *The logarithm of the square root of the product of two numbers is the half sum of their logarithms to the same base.* [th. 6

E.g., $\log_{A}\sqrt{(B \cdot C)} = \frac{1}{2}(\log_{A} B + \log_{A} C)$.

THEOR. 8. *If the logarithm of any same number be taken to two different bases, the first logarithm equals the product of the second logarithm into the logarithm of the second base taken to the first base, and* vice versa.

Let N be any number, A, B two bases;

then will $\log_A N = \log_B N \cdot \log_A B$, and $\log_B N = \log_A N \cdot \log_B A$.

For let $y = \log_B N$; [df. log

then $\because$ $N = B^y$,

and $\log_A N = y \cdot \log_A B$, [th. 7

$\therefore$ $\log_A N = \log_B N \cdot \log_A B$. Q.E.D.

So, $\log_B N = \log_A N \cdot \log_B A$. Q.E.D.

COR. 1. $log_A B \cdot log_B C \cdot log_C D = log_A D$.

$log_A B \cdot log_B C \cdot log_C D \cdots log_K L = log_A L$.

COR. 2. *The logarithms of two numbers, each taken to the other number as base, are reciprocals.*

For let $N = A$;

then $\log_A B \cdot \log_B A = \log_A A = 1$. Q.E.D.

COR. 3. $log_A B \cdot log_B C \cdot log_C A = 1$;

$log_A B \cdot log_B C \cdot log_C D \cdots log_K A = 1$.

NOTE. The reader will observe that the bases and numbers run in cyclic order:

$$\begin{tikzcd} A \arrow[d] & D \arrow[l] \\ B \arrow[r] & C \arrow[u] \end{tikzcd}$$

COR. 4. *The modulus of any system of logarithms is the logarithm, in that system, of the Napierian base* e. [VIII. th. 15 nt.

Let A be the base of any system of logarithms, and M_A the modulus;

then $\because$ $\log_A x = \log_A e \cdot \log_e x$, [th. 8

wherein $\log_A e$ is a constant, independent of x,

$\therefore$ $D_x \log_A x = \log_A e \cdot D_x \log_e x$,

i.e., $\frac{M_A}{x} = \log_A e \cdot \frac{1}{x}$, [VII. th. 16

$\therefore$ $M_A = \log_A e$. Q.E.D.

E.g., $M_{10} = \log_{10} e = \log_{10} 2.71828 \cdots$ [

$= .4342944 \cdots$.

§ 2. SPECIAL PROPERTIES, BASE 10.

The logarithm of an exact power of 10 is an integer. [df.log

E.g., of ..., 1000, 100, 10, 1, .1, .01, .001, ...
the logarithms to base 10 are
..., $^+3$, $^+2$, $^+1$, 0, $^-1$, $^-2$, $^-3$, ...

But of any other number the logarithm is fractional or incommensurable, and consists of a whole number, the *characteristic*, and an endless decimal, the *mantissa*. [VIII. § 4 df. incom. pwr.

As a matter of convenience the mantissa is always taken positive; and the characteristic is the exponent, positive or negative, of the integral power of 10 next below the given number.

A negative characteristic is indicated by the sign — above it.

E.g., of the numbers
2000, 20, .2, .002,
the logarithms to base 10 are
3.30103 ..., 1.30103 ..., $\bar{1}$.30103 ..., $\bar{3}$.30103 ...,
whose characteristics are 3, 1, $\bar{1}$, $\bar{3}$,
and whose common mantissa is $^+$.30103 ....

THEOR. 9. *If a number be multiplied (or divided) by any integral power of* 10, *the logarithm of the product (or quotient) and the logarithm of the number have the same mantissa.*

For ∵ the logarithm of a product is the sum of the logarithms of its factors. [th. 6

and ∵ the logarithm of the multiplier is integral, [hyp.

∴ the mantissa of the sum is identical with the mantissa of the logarithm of the multiplicand. Q.E.D.

So, if a number be divided by an integral power of 10.

COR. *For all numbers that consist of the same significant figures in the same order, the mantissa of the logarithm is constant, but the characteristic changes with the position of the decimal point in the number.*

E.g., of the numbers
79500, 795, 7.95, .0795, .000795,
the logarithms to base 10 are
4.9004, 2.9004, 0.9004, $\bar{2}$.9004, $\bar{4}$.9004.

§ 3. COMPUTATION OF LOGARITHMS.

Prob. 1. To compute the logarithm of a number to a given base.

FIRST METHOD, BY CONTINUED FRACTIONS.

Form the exponential equation, $A^x = N$, *wherein* N *is the number,* A *the base, and* x *the logarithm sought.* [df. log

By trial find two consecutive integers, x' *and* $x'+1$, *between which* x *lies, and write* $x = x' + y^{-1}$, *wherein* x' *is known and* y^{-1} *is some positive number less than unity.*

In the equation $A^x = N$, *replace* x *by* $x' + y^{-1}$, *giving* $A^{x'+y^{-1}} = N$, *and divide both members by* $A^{x'}$, *giving* $A^{\frac{1}{y}} = N : A^{x'}, \equiv N'$, *say.*

Raise both members of the equation $A^{\frac{1}{y}} = N'$ *to the* y*th power, giving* $A = N'^{y}$.

By trial find two consecutive integers, y' *and* $y'+1$, *between which* y *lies, write* $y = y' + z^{-1}$, *and so on, as above.*

Then $x = x' + \dfrac{1}{y} = x' + \dfrac{1}{y' + \dfrac{1}{z}} = x' + \dfrac{1}{y' + \dfrac{1}{z' + \cdots}}$

and the convergents, which approach x *as their limit, are:*

$$x', \quad \frac{x'y'+1}{y'}, \quad \frac{x'y'z' + z' + x'}{y'z'+1}, \quad \cdots.$$

E.g., given $10^x = 5$, to find x, *i.e.* to find $\log_{10} 5$.

Put $x = 0 + y^{-1}$,

then $\because$ $10^{\frac{1}{y}} = 5$, $\qquad 5^y = 10$, $y = 1 + \dfrac{1}{z}$;

$5^{1+\frac{1}{z}} = 10$, $\quad 5^{\frac{1}{z}} = 2$, $\qquad 2^z = 5$, $z = 2 + \dfrac{1}{r}$;

$2^{2+\frac{1}{r}} = 5$, $\quad 2^{\frac{1}{r}} = \dfrac{5}{4}$, $\qquad \left(\dfrac{5}{4}\right)^r = 2$, $r = 3 + \dfrac{1}{s}$;

$\left(\dfrac{5}{4}\right)^{3+\frac{1}{s}} = 2$, $\left(\dfrac{5}{4}\right)^{\frac{1}{s}} = \dfrac{128}{125}$, $\left(\dfrac{128}{125}\right)^s = \dfrac{5}{4}$, $s = 9 + \dfrac{1}{t}$;

and so on.

$$\therefore x = \frac{1}{1+\dfrac{1}{z}} = \frac{1}{1+\dfrac{1}{2+\dfrac{1}{r}}} = \frac{1}{1+\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{s}}}} = \frac{1}{1+\dfrac{1}{2+\dfrac{1}{3+\dfrac{1}{9+\cdots}}}},$$

and the convergents are

$$1, \quad \frac{2}{3}, \quad \frac{7}{10}, \quad \frac{65}{93}, \quad \cdots.$$

These convergents are alternately too large and too small; but their errors are respectively less than

$$\frac{1}{3};\ \frac{1}{3\cdot 10}=\frac{1}{30};\ \frac{1}{10\cdot 93}=\frac{1}{930};\ \frac{1}{93\cdot \text{next denominator}},$$

which denominator is not less than $93+10$, $=103$; [VI. ths. 1, 2

$\therefore\ \frac{65}{93}$, $=.69892\cdots$, is too small, and differs from the true value by less than $\frac{1}{9579}$.

The true logarithm of five to seven decimal places, as shown by the tables, is .6989700, so that $\frac{65}{93}$ actually differs from it by less than half of one ten-thousandth.

So, $\log_{10}2 = \log_{10}10 - \log_{10}5 = 1 - .69897 = .30103$.

So, $\log 4 = 2\cdot\log 2 = .60206$; $\log 8 = 3\cdot\log 2 = .90309$;

$\log 625 = 4\cdot\log 5 = 2.79588$; $\log\frac{4}{5} = \log 4 - \log 5 = \bar{1}.90309$.

SECOND METHOD, BY SUCCESSIVE SQUARE ROOTS OF PRODUCTS.

Take two numbers whose logarithms are known, the one greater and the other less than the given number.

Find the square root of their product and the logarithm of this root, the half sum of their logarithms. [th. 10

Multiply this root by whichever of the two numbers lies at the other side of the given number, and find the square root of the product, and the half sum of the logarithms of the factors; and so on.

E.g., to find the logarithm of 5 to the base 10:

Take 10 whose logarithm is 1, and 1 whose logarithm is 0;

		Number.		Logarithm.
then	$\sqrt{(10\times 1)}$	$=3.16227766$;	$\frac{1}{2}(1+0)$	$=.5$
	$\sqrt{(10\times 3.16227766)}$	$=5.62341325$;	$\frac{1}{2}(1+.5)$	$=.75$
	$\sqrt{(3.16228\times 5.62341)}$	$=4.21696535$;	$\frac{1}{2}(.5+.75)$	$=.625$
	$\sqrt{(5.62341\times 4.21697)}$	$=4.86967671$;	$\frac{1}{2}(.75+.625)$	$=.6875$
	$\sqrt{(5.62341\times 4.86968)}$	$=5.23299218$;	$\frac{1}{2}(.75+.6875)$	$=.71875$
	$\sqrt{(4.86968\times 5.23299)}$	$=5.04806762$;	$\frac{1}{2}(.6875+.71875)$	$=.70312$
	$\sqrt{(4.86968\times 5.04807)}$	$=4.95807276$;	$\frac{1}{2}(.6875+.703125)$	$=.69531$
	$\sqrt{(4.95807\times 5.04807)}$	$=5.00028680$;	$\frac{1}{2}(.69531+.70312)$	$=.69921$
	$\sqrt{(4.95807\times 5.00029)}$	$=4.97709632$;	$\frac{1}{2}(.69921+.69531)$	$=.69726$
			$\frac{1}{2}(.69921+.69726)$	$=.69823$

§ 4. TABLES OF LOGARITHMS.

If for successive equidistant values of a variable the corresponding values of a function of this variable be arranged in order, the function is *tabulated;* the variable is the *argument* of the table [I. § 13] and the successive values of the function are the *tabular numbers.* The values of the argument are commonly placed in the margin of the table.

If the logarithms, to any one base, of the successive integers from 1 to a given number, say 1000, or 10000, be arranged for ready reference, they form a *table of logarithms.* Such tables are in use to three places of decimals, to four, five, six, seven, and even ten, twenty, or more places.

In general, the greater the number of decimal places, the greater the accuracy, and the greater the labor of using the tables. For the ordinary use of the engineer, navigator, chemist, or actuary, four- or five-place tables are sufficient; but most refined computations in Astronomy or Geodesy require at least seven-place tables.

Most logarithmic tables are arranged on the same general plan as the four-place table given on pp. 248, 249. This table gives the mantissa only; the computer can readily supply the characteristic. To save space, the first two figures of each argument are printed at the left of the page, and the third figure at the top of the page over the corresponding logarithm.

To save time, labor, and injury to the eyes, the computer should use a well-arranged table, and then train himself to certain habits. The best tables have the numbers grouped by spaces, or by light and heavy lines, into blocks of three or five lines, and three or five columns, corresponding to the right-hand figures of the arguments of the table. The usual patterns are

|0|1 2 3|4 5 6|7 8 9|0|1 2 3| ⋯ for three-line blocks,

and |0 1 2 3 4|5 6 7 8 9|0 1 2 3 4|⋯ for five-line blocks,

as in the table on pp. 248, 249. Instead of tracing single lines of figures across the page and down the column, the computer should learn to guide himself by correspondences of position in the blocks.

§ 5. OPERATIONS WITH COMMON LOGARITHMS.

PROB. 2. TO TAKE OUT THE LOGARITHM OF A GIVEN NUMBER.

(a) *One, two, or three significant figures.*

If the number have one significant figure, annex two zeros; if two significant figures, annex one zero; for the mantissa write the four figures that lie opposite the first two figures and under the third figure, and for the characteristic write the exponent of the power of 10 *next below the given number.*

E.g., $\log 567 = 2.7536$; $\log 5.6 = 0.7482$; $\log .05 = \bar{2}.6990$;

If a number have more than three significant figures, the mantissa of its logarithm is not found in the table, but lies between two tabular mantissas whose arguments are two three-figure numbers next larger and next smaller than the given number. [th. 3

E.g., mantissa log 500.6 lies between .6990, .6998,
i.e., between mantissa logs 500, 501.

(b) *Four or more significant figures.*

Find the mantissa of the logarithm of the first three figures as above; subtract this mantissa from the next larger tabular mantissa, and take such part of the difference as the remaining figures are of a unit having the rank of the third figure; add this product, as a correction, to the mantissa of the first three figures.

E.g., to find log 500.6;

then $\because \log 500 = 2.6990, \ \log 501 = 2.6998,$ [tables
and $\log 501 - \log 500 = .0008, \ 500.6 - 500 = .6,$
$\therefore \log 500.6 = 2.6990 + .6 \text{ of } .0008 = 2.6995.$

NOTE 1. If the given number lie nearer the larger of the two arguments, its mantissa is easiest found by subtracting from the larger of the two tabular mantissas such part of their difference as the excess of the larger argument over the given number is of a unit having the rank of the third figure.

E.g., to find mantissa log 500.6;

then $\because$ mantissa logs 500, 501 = .6990, .6998, [tables.
and $\because \log 501 - \log 500 = .0008, \ 501 - 500.6 = .4,$
$\therefore$ mantissa log 500.6 = .6998 − .4 of .0008 = .6995.

NOTE 2. The rule for *interpolating* or applying the correction rests upon a property which logarithms have in common with most other functions, and which the reader may observe for himself if he will examine the table carefully, viz.: that the differences of logarithms are very nearly proportional to the differences of their numbers when these differences are small. They are not exactly proportional, but the error is so small as to be inappreciable when using a four-place table. The seven-place tables give the logarithms of all five-figure numbers, and the errors for the sixth, seventh, and eighth figures, as far as due to this cause, are inappreciable. So the rule above given "for applying the correction" is universal.

NOTE 3. The computer should train himself to find the correction and add it to the tabular mantissa (or subtract it) mentally, and to write down only the final result.

To aid in this mental computation, small tables of *proportional parts* are often printed at the side of the principal table. Two forms of such tablets are here shown: the first most accurate, and the other of easiest use.

	19	18
1	1.9	1.8
2	3.8	3.6
3	5.7	5.4
4	7.6	7.2
5	9.5	9.0
6	11.4	10.8
7	13.3	12.6
8	15.2	14.4
9	17.1	16.2

or

	19	18
1	2	2
2	4	4
3	6	5
4	8	7
5	10	9
6	11	11
7	13	13
8	15	14
9	17	16

E.g., to find mantissa log 22674;

then $\because$ $\log 227 - \log 226 = .3560 - .3541 = .0019$,

$\therefore$ the correction to be added to .3541 is

$.7$ of $.0019 + .04$ of $.0019$; and is found thus:

opposite 7 find	13.3	or	13	.3541
opposite 4 find	.8		1	+14
Add; the correction is	14		14	giving .3555

Or $\because$ $22700 - 22674 = 26$,

$\therefore$ the correction to be subtracted from .3560 is

$.2$ of $.0019 + .06$ of $.0019$; and is found thus:

opposite 2 find	3.8	or	4	.3560
opposite 6 find	1.1		1	−5
Add; the correction is	5		5	.3555

PROB. 3. TO FIND A NUMBER FROM ITS LOGARITHM.

(*a*) *The mantissa found in the table.*

Write down the two figures opposite to the given mantissa in the left-hand column, and following them the figure at the top of the column in which the mantissa is found.

Place the decimal point so that the number shall be next above that power of 10 *whose exponent is the given characteristic.*

E.g., $\log^{-1} 2.7536 = 567$; $\log^{-1} 0.7482 = 5.6$; $\log^{-1} \bar{2}.6990 = .05$.

(*b*) *The mantissa not found in the table.*

Take out the first three figures for the tabular mantissa next less, as above; from the given mantissa subtract this tabular mantissa, and divide the difference by the difference between the tabular mantissa next less and that next greater.

Annex the quotient to the three figures first found.

Place the decimal point as above.

E.g., to find $\log^{-1} 2.6995$.

then $\because$ $\log^{-1} 2.6990 = 500$, $\log^{-1} 2.6998 = 501$, [tables

and $\because$ $2.6995 - 2.6990 = .0005$, $2.6998 - 2.6990 = .0008$;

$\therefore$ the number sought is $500 + (.0005 : .0008)$, $= 500.6$.

NOTE 1. The process is but the inverse of that for taking out logarithms, and the reason of the rule is the same for both.

This four-place table allows only one-figure corrections, and so gives only four-figure numbers. In general, an n-place table gives n-figure numbers; but sometimes, when the mantissa is large, the nth figure may be two or three units in error, and then the number is approximate only for $n-1$ figures [V. § 5].

NOTE 2. If the given mantissa lie nearer the larger of the two tabular mantissas, the correction may be applied to the larger argument by subtraction.

E.g., to find $\log^{-1} .3555$;

then $\because$ the next tabular mantissas .3541, .3560 differ by .0019, and correspond to 226, 227, as arguments,

and $\because$ $.3555 - .3541 = .0014$, $.3560 - .3555 = .0005$,

$\therefore$ the number sought is $226 + \frac{14}{19}$, or $227 - \frac{5}{19}$, $= 226.74$.

If the tablets of proportional parts be used, the work, written out, appears as follows:

14	226	or	5	227
13.3	+.7		3.8	−.2
.7	+ 4		1.2	− 6
.8	226.74		1.1	226.74

PROB. 4. TO FIND, BY ONE OPERATION, THE ALGEBRAIC SUM OF SEVERAL LOGARITHMS.

Arrange the logarithms vertically, and take the algebraic sum of each column of digits, beginning at the right and carrying as in ordinary addition; if this sum for any column be negative, make it positive by adding one or more tens to it and subtract as many units from the next column.

E.g., to find the algebraic sum in the margin, adding upward, the computer says:

	3.1037
	-0.6986
9, 7, 16, 10, 17,	$+\bar{2}.2409$
1, 3, $^-6$, $^-14$, $^-11$, 9, 2 off,	$-\bar{2}.5892$
$^-2$, 3, $^-5$, $^-1$, $^-10$, 0, 1 off,	$+\bar{1}.2529$
$^-1$, 1, $^-4$, $^-2$, $^-8$, $^-7$, 3, 1 off,	$=1.3097$
$^-1$, $^-2$, 0, $^-2$, 1,	

and, adding downward, for a check, he says:

7, 1, 10, 8, 17,

1, 4, $^-4$, $^-13$, $^-11$, 9, 2 off, and so on.

PROB. 5. TO DIVIDE A LOGARITHM WHOSE CHARACTERISTIC IS NEGATIVE.

Write down the number of times the divisor goes into that multiple of itself which is equal to, or next less than, the negative characteristic; carry on the positive remainder to the mantissa, and divide.

E.g., $\bar{4}.1234 : 3 = (^-6 + 2.1234) : 3 = \bar{2}.7078$.

So, $\bar{3}.4770 \cdot \frac{3}{2} = \bar{8}.4310 : 2 = \bar{4}.2155$.

PROB. 6. TO AVOID NEGATIVE CHARACTERISTICS.

Modify the logarithms by adding 10 *to their characteristics when negative; use the sums, differences, or exact multiples of the modified logarithms where the subject-matter is such that the computer cannot mistake the general magnitude of the results.*

To divide a modified logarithm, add such a multiple of 10 *as will make the modified logarithm exceed the true logarithm by* 10 *times the divisor; then divide.*

E.g., if $\log a = \bar{2}.3010$, $\log b = \bar{1}.4771$, to find $\log(a^{\frac{2}{5}} b^{-\frac{3}{5}})$,
$= \frac{1}{5}(2\log a - 3\log b)$.

BY TRUE LOGARITHMS.	BY MODIFIED LOGARITHMS.
$\bar{2}.3010 \cdot 2 = \bar{4}.6020$	$8.3010 \cdot 2 = 6.6020$
$\bar{1}.4771 \cdot 3 = \underline{\bar{2}.4313}$	$9.4771 \cdot 3 = \underline{8.4313}$
$5)\underline{\bar{2}.1707}$	$5)\underline{8.1707}$
$\bar{1}.6341$	9.6341

At each step of the work with modified logarithms, any tens in the characteristics are rejected, or tens, if necessary, are added, so as to keep the characteristics between 0 and 9 inclusive. Before dividing by 5, in the example just above, 4 tens were added, making the dividend 48.1707.

NOTE. The *arithmetical complement* of the logarithm of a number is the modified logarithm of the reciprocal of the number. It is got by subtracting the given logarithm, modified, if necessary, from 10; it may be read from the table by subtracting each figure from 9, beginning with the characteristic and ending with the last significant figure but one, subtracting the last significant figure from 10, and annexing as many zeros as the given logarithm ends with. The arithmetical complement of the arithmetical complement is the original logarithm.

E.g., ar-com $3.4908000 = 6.5092000$, and conversely.

In any algebraic sum, a subtractive logarithm can be replaced by its arithmetical complement taken additively. In most cases, however, the method of prob. 4 appears preferable.

E.g., in the example under prob. 4, the terms -0.6986. $-\bar{2}.5892$ might be replaced by 9.3014, 1.4108.

PROB. 7. TO COMPUTE BY LOGARITHMS THE PRODUCTS, QUOTIENTS, POWERS, AND ROOTS OF NUMBERS.

1. *For a product: add the logarithms of the factors, and take out the antilogarithms of the sum.*

2. *For a quotient: from the logarithm of the dividend subtract that of the divisor, and take out the antilogarithm.*

3. *For a power: multiply the logarithm of the base by the exponent of the power sought, and take out the antilogarithm.*

4. *For a root: divide the logarithm of the base by the root-index, and take out the antilogarithm.*

E.g., to find the value of $(.01519 \cdot 6.318 : 7.254)^{\frac{3}{2}}$:

NUMBERS.	LOGARITHMS.
.01519	$\bar{2}.1815$
$\times 6.318$	$+0.8006$
$\div 7.254$	-0.8605
	$\bar{2}.1216 \times \frac{3}{2}$
and the number sought is 0.001522.	$\bar{3}.1824$

NOTE. Not only simple operations, as in the above example, but complex operations, can be performed by logarithms. Sometimes the expression whose value is sought must first be prepared by factoring.

E.g., to find the value of $\sqrt{(h^2 - b^2)}$, wherein h, b are any given numbers and may represent the lengths of the hypothenuse and base of a right triangle:

then $\quad \sqrt{(h^2 - b^2)} = \log^{-1} \frac{1}{2}(\log \overline{h+b} + \log \overline{h-b})$.

PROB. 8. TO SOLVE THE EXPONENTIAL EQUATION $A^x = B$.

Divide the logarithm of B *by the logarithm of the base* A *of the exponential: the quotient is* X, *the exponent sought.*

For $\because A^x = B$,

$\therefore x \log A = \log B$,

$\therefore x = \log B : \log A$. Q.E.D.

PROB. 9. TO ESTIMATE THE AMOUNT OF POSSIBLE ERROR IN A LOGARITHM OR ANTILOGARITHM GOT FROM THE TABLE, AND IN THE SOLUTIONS OF PROBS. 7, 8:

Let p *be the number of decimal places in the table used;* A', B', $\cdots X'$, $(A^m B^n \cdots)'$, *the number of units of their last decimal places contained in* $A, B, \cdots X$, $A^m B^n \cdots$; $\alpha, \beta, \cdots$, *the possible relative errors, all taken positive, of* A, B, $\cdots$: *then*

(*a*) *Poss. err. log* $X = 10^{-p} + .43$ *poss. rel. err.* X.

(*b*) *Poss. rel. err.* $X = 1 : 2X' + 2.3 \cdot$ *poss. err. log* X.

(*c*) *Poss. rel. err.* $A^m B^n \cdots$ [*in pr.* 7]

$$= 1 : 2\,(A^m B^n \cdots)' + 2.3\,(m + n + \cdots) \cdot 10^{-p} + (m\alpha + n\beta + \cdots).$$

(*d*) *Poss. rel. err.* X [*in pr.* 8]

$$= \frac{1}{2X'} + \frac{10^{-p} + .43\alpha}{\log A} + \frac{10^{-p} + .43\beta}{\log B}.$$

For $\because D_x \log_{10} X = M_{10}\frac{1}{X}$, [VIII. th. 15, A = 10

$$\therefore M_{10} = X \cdot D_x \log_{10} X \doteq \frac{X \cdot \text{inc} \log X}{\text{inc}\, X} = .43, \text{ [table logs}$$

$$\therefore \frac{\text{inc} \log X}{\text{inc}\, X} \doteq \frac{.43}{X},$$

i.e., $$\text{inc} \log X \doteq .43\frac{\text{inc}\, X}{X},$$

and $$\frac{\text{inc}\, X}{X} \doteq \frac{\text{inc} \log X}{.43} = 2.3 \text{ inc.} \log X :$$

(*a*) $\because$ log X, as got from X by P-place logarithm-tables, has a possible error composed of:

two possible half-units in *p*th decimal place, from the omitted decimals of the printed logarithm and of the correction for interpolation,

and an increment or error, $\doteq .43 \cdot \dfrac{\text{inc. or err. of } X}{X}$; [above

$$\therefore \text{poss. err.} \log X \doteq (\tfrac{1}{2} + \tfrac{1}{2})10^{-p} + .43 \cdot \frac{\text{err. } X}{X}$$

$$= 10^{-p} + .43 \cdot \text{poss. rel. err. } X. \quad \text{Q.E.D.}$$

(*b*) $\because$ X, as got from log X by the same table, has a possible error composed of:

a possible half-unit in last decimal place, for the omitted decimals,

and an increment or error, $\doteq 2.3 \cdot \mathrm{x} \cdot \text{inc.} \log \mathrm{x}$,

$\therefore$ poss. err. $\mathrm{x} \doteq \frac{1}{2}$ in last decimal place of $\mathrm{x} + 2.3 \cdot \mathrm{x} \cdot$ poss. err. $\log \mathrm{x}$,

$\therefore$ poss. rel. err. $\mathrm{x} \doteq 1 : 2\mathrm{x}' + 2.3 \cdot$ poss. err. $\log \mathrm{x}$. Q.E.D.

(*c*) $\because$ poss. err. $\log \mathrm{A} = 10^{-p} + .43\,a$, [(*a*)

$\therefore$ poss. err. $\log \mathrm{A}^m = m\,(10^{-p} + .43\,a$

So, poss. err. $\log \mathrm{B}^n = n\,(10^{-p} + .43\,\beta)$, $\cdots$,

$\therefore$ poss. err. $\log (\mathrm{A}^m \mathrm{B}^n \cdots)$

$= (m + n + \cdots)\,10^{-p} + .43\,(ma + n\beta + \cdots)$,

$\therefore$ poss. rel. err. $(\mathrm{A}^m \mathrm{B}^n \cdots)$

$= 1 : 2\mathrm{x}' + 2.3[(m+n+\cdots)10^{-p} + .43(ma + n\beta + \cdots)]$,

wherein $\mathrm{x} \equiv \mathrm{A}^m \mathrm{B}^n \cdots$;

$\therefore$ poss. rel. err. $(\mathrm{A}^m \mathrm{B}^n \cdots)$

$= 1 : 2\mathrm{x}' + 2.3\,(m + n + \cdots)\,10^{-p} + (ma + n\,? + \cdots)$.

Q.E.D.

(*d*) $\because$ $\mathrm{x} = \log \mathrm{B} : \log \mathrm{A}$,

$\therefore$ poss. rel. err. x = poss. rel. err. from omitted decimals of x + poss. rel. err. $\log \mathrm{A}$ + poss. rel. err. $\log \mathrm{B}$ [V. th. 5 cr. 3

$$= \frac{1}{2\mathrm{x}'} + \frac{10^{-p} + .43\,a}{\log \mathrm{A}} + \frac{10^{-p} + .43\,\beta}{\log \mathrm{B}}. \quad \text{Q.E.D.} \quad [(a)$$

NOTE. If in (*d*) the divisions $\log \mathrm{B} : \log \mathrm{A}$ be performed by logarithms,

then $\because$ $\log \mathrm{x} = \log\ \log \mathrm{B} - \log\ \log \mathrm{A}$,

$\therefore$ poss. err. $\log \mathrm{x}$ = poss. err. log·log A + poss. err. log·log B

$= 10^{-p} + .43$ poss. rel. err. $\log \mathrm{A}$

$+ 10^{-p} + .43$ poss. rel. err. $\log \mathrm{B}$ [(*a*)

$$= 2 \cdot 10^{-p} + .43\left(\frac{10^{-p} + .43\,a}{\log \mathrm{A}} + \frac{10^{-p} + .43\,\beta}{\log \mathrm{B}}\right),$$

$\therefore$ poss. rel. err. x, $= \dfrac{1}{2\mathrm{x}'} + 2.3$ poss. err. $\log \mathrm{x}$ [(*b*)

$$= \frac{1}{2\mathrm{x}'} + 4.6 \cdot 10^{-p} + \frac{10^{-p} + .43\,a}{\log \mathrm{A}} + \frac{10^{-p} + .43\,\beta}{\log \mathrm{B}},$$

which differs from the former result only by the term $4.6 \cdot 10^{-p}$ arising from the omitted decimals of the table used in performing the division, and obtainable also from (*c*) by making $m = n = 1$, $a = \beta = 0$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

§ 6. EXAMPLES.

1. What is the logarithm of 144:
   to base $2\sqrt{3}$? to base $2\sqrt[3]{12}$? to base $(2\sqrt[3]{12})^{-1}$?
2. What is the characteristic of:
   $\log_2 7$? $\log_7 2$? $\log_3 21$? $\log_{21} 3$? $\log_{\frac{1}{3}} 21$? $\log_{\frac{1}{3}} 21^{-1}$?
3. Find $\log_5 3125$; $\log_7 343^{-1}$; $\log_{\frac{1}{3}} 81$; $\log_{\frac{1}{7}} 343$; $\log_{\frac{1}{7}} 343^{-1}$.

§ 3, PROB. 1.

4. By continued fractions derive the logarithms, to base 10, of 3 and 7 to four decimal places.
   Thence find the logarithms of:
   9, 2.7, .81, 70, 4.9, 343, 21, 63, .441, $.7^{-1}$, 18.9^{-1}.

§ 5, PROB. 2.

5. From the table take out the logarithms of:
   12, 120, 123, 124, 123.4, 1.234, 12350, .001235.

§ 5, PROB. 3.

6. From the table find the antilogarithms of:
   1.0792, 2.0792, 2.0899, 2.0934, 2.0913, 0.0913, $\bar{4}.0917$.

§ 5, PROBS. 4–8.

7. By logarithms find the values of:
   $$\frac{2^2 5^3 85^2}{3^2 7^3},\quad \frac{\sqrt{(97^2-9^2)}}{81\cdot\sqrt[3]{572}},\quad \frac{\sqrt{12}\cdot\sqrt[3]{65}}{\sqrt{5}\cdot\sqrt[7]{.\dot{1}\dot{8}}},\quad \frac{\sqrt[3]{83.64}\times 39.56^2}{.08145^2\times\sqrt[5]{1.968}}.$$
8. From the logarithm of 2 find the number of digits in:
   2^{64}, 2^{500}, 5^{100}, 20^8, 160^{10}, 25^{25}, $6.25^{6.25}$, 25^{-4}, 50^{-50}.
9. By logarithms multiply 575.25 by 1.06^{20}; by 1.03^{40}; by 1.015^{80}.
10. By logarithms find $\sqrt[3]{1000}$, $\sqrt[5]{.00010098}$, $\sqrt[7]{.0000000037591}$.
11. What power is 2 of 1.05? 3 of 1.04? 4 of 1.03? 5 of 1.02?
12. If the number of births per year be 1 in 45, and of deaths 1 in 60, in how many years will the population double, taking no account of other sources of increase or decrease?

§ 5, PROB. 9.

13. Find the possible error in each of the examples in Nos. 7–12.

X. IMAGINARIES.

☞ BEFORE taking up this chapter, the reader may refer to what is said of numbers in I. § 1 and of negatives in I. § 3; and particularly the note at the end of I. § 3. He will observe that, for some kinds of quantity, negatives as well as fractions are impossible. He may not be surprised, therefore, to learn that, even if the operation denoted by imaginary numbers can be conveniently performed upon only one kind of magnitude, they have most of the properties of real numbers and play an important part in algebra. These operations can, however, be performed, though less simply, upon all kinds of magnitude, as appears in chapter XV.

§ 1. DEFINITIONS AND GRAPHIC REPRESENTATION.

IN measuring any thing some unit of like kind is first assumed, and the relation the thing measured bears to this unit, both as to magnitude and as to sense or quality, is expressed by a number [I. § 1]. Conversely, this number expresses that operation which must be performed upon the unit to produce the thing: the unit being then the operand, the number the operator, and the thing the result of the operation.

POSITIVE AND NEGATIVE NUMBERS.

In the method of graphic representation of numbers here described, a finite straight line pointing in an assumed direction is chosen as the concrete unit; and the relation that any straight line pointing in the $\left\{\begin{matrix}\text{same}\\ \text{opposite}\end{matrix}\right.$ direction has to this unit is expressed by a $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ number.

If the reader so place himself before the unit that to him it becomes horizontal and points to the right, $\longrightarrow$, then any horizontal line pointing to the right, $\longrightarrow$, has its length and direction in terms of the unit line expressed by a positive number. If the line be taken up and reversed, so that it is still horizontal but points to the left, $\longleftarrow$, then

the relation of its length and direction to the unit is expressed by a negative number. The length remains as before; but the quality, or direction, is reversed.

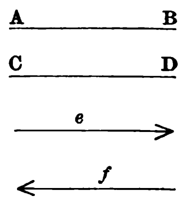
VECTORS.

A *vector*, or *directed right line*, is any line whose length and direction are considered, but not its location. Its two extremities are distinguished from each other as its *initial point* and its *terminal point*. Its *direction* is the direction of the terminal point from the initial point, and would be reversed if these points were interchanged.

As the name implies, a vector may be regarded as the representative of the operation of carrying a particle from its initial point to its terminal point.

The direction of a vector may be designated by the order in which its two extremities are named, or by an arrow-head.

Equal
Opposite *vectors* are those having the same length and
{ the same direction.
{ opposite directions.



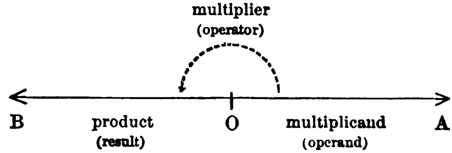
E.g., the vectors AB, CD, e are equal to each other, but are opposite to the vectors BA, DC, f.

A *vector quantity* is any concrete quantity whose magnitude and direction only are considered, and which is naturally represented by a measured and directed right line or vector.

E.g., the direction and velocity or force of the wind, or of an electric current, is a vector quantity, and may be represented by an arrow.

COMPLETE REVERSALS.

When the operand is a vector, the operation of multiplying it by $^{-}1$ consists in reversing its quality or direction, and is exhibited thus:



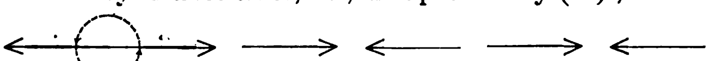
So, even if the operand be not a vector, yet if it have a reversible quality, the vector OA may still be taken as the *representative* of the operand; and, since to multiply the operand by $^{-}1$ is simply to change its quality into the opposite quality, this multiplication is *graphically represented* by the reversal AOB, while the result is represented by the vector OB. Hence, in what follows, the vectors used may be either the actual operands and results, or merely their representatives.

If now there be a continuous rotary motion, as with a spoke of a wheel, the direction or quality of the vector OA is alternately reversed and restored:

E.g., a half revolution, one reversal, is multiplication by $^{-}1$.

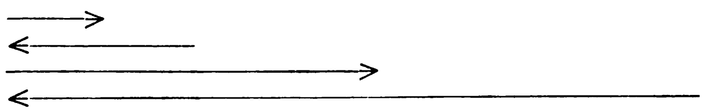
So, a whole revolution, two reversals, is multiplication by $^{-}1$ twice, *i.e.*, multiplication by $(^{-}1)^2$, $=^{+}1$.

So, a revolution and a half, three reversals, is multiplication by $^{-}1$ three times, *i.e.*, multiplication by $(^{-}1)^3$, $=^{-}1$.



no rev'l. 1 rev'l. 2 rev'l. 3 rev'l.

So, multiplying a vector by $^{-}2$ doubles the vector and reverses it; multiplying by $(^{-}2)^2$ doubles it twice and reverses it twice; and so on; and the like is true whether the operand be a vector or not.



By such multiplication two distinct effects are produced: the one quantitative, the ordinary multiplication of arithmetic, which consists in stretching the line multiplied; the other qualitative, which consists in reversing the direction of the line.

Every such multiplier or number may be regarded as itself the product of two factors: its *tensor*, the quantitative or stretching factor; and its *versor*, the qualitative or turning factor.

If the tensor $\left\{ \begin{matrix} > \\ < \end{matrix} \right. 1$, its effect is to $\left\{ \begin{matrix} \text{lengthen} \\ \text{shorten} \end{matrix} \right.$ the multiplicand.

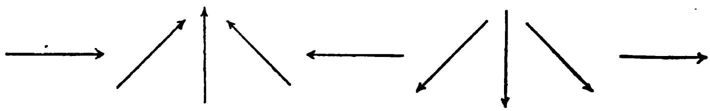
E.g., the number $^{-}3$ is the product of tensor 3 and versor $^{-}1$.

So, the number $^{\pm}\frac{2}{3}$ is the product of tensor $\frac{2}{3}$ and versor $^{\pm}1$.

PARTIAL REVERSALS.—IMAGINARIES.

But during its rotation the line has filled various intermediate positions wherein the numbers expressing its relation to the unit were neither purely positive nor purely negative numbers:

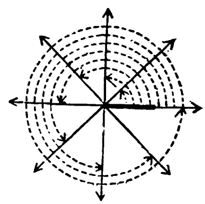
E.g., in the positions



its relations or *ratios* to a unit $\longrightarrow$ are:

$^{+}2$, intermediate, $^{-}2$, intermediate, $^{+}2$,

and are represented thus:



These intermediate numbers are *imaginaries*, or *imaginary numbers*, and may be defined as numbers, not 0, that are neither purely positive nor purely negative.

By way of distinction, positive and negative numbers, the ordinary numbers of arithmetic and algebra, are *real numbers.*

It appears later [XIII.] that every imaginary number of ordinary algebra involves an even root of a negative, and arises from an attempt to violate a condition of maximum or minimum: as in seeking the base of a right triangle whose height shall exceed the hypothenuse.

The square root of a negative real number is a *pure imaginary;* all other imaginaries are *complexes.*

E.g., the value of $\sqrt{^{-}4}$ is not $^{+}2$, whose square is $^{+}4$, nor $^{-}2$, whose square is also $^{+}4$; it is something different from either, and intermediate between them in character.

So, most roots, whether odd or even, and whether of positive or negative bases, have imaginary values, as appears later.

THE SYMBOL $\sqrt{^{-}1}$.

The symbol $\sqrt{^{-}1}$ denotes a number whose square is $^{-}1$: *i.e.*, it is a number such that unit $\times \sqrt{^{-}1} \times \sqrt{^{-}1} =$ unit $\times\, ^{-}1$. [I. § 10 df. root

Hence, whatever meaning is given to multiplication by $^{-}1$, a

consequent meaning must be given to multiplication by $\sqrt{{}^{-}1}$ such that two successive multiplications by $\sqrt{{}^{-}1}$ shall produce the same result as one multiplication by ${}^{-}1$.

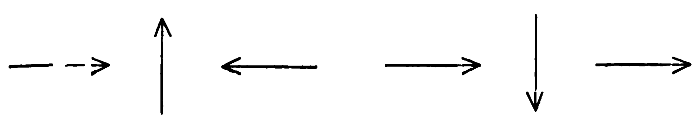
If the unit be a horizontal line pointing to the right, then the product, unit $\times \sqrt{{}^{-}1}$, is a vertical line of unit length pointing either upward or downward; for if the horizontal unit-line be first revolved to a perpendicular either way, then the same amount of further rotation will bring it to the opposite horizontal position. Here multiplication by $\sqrt{{}^{-}1}$ consists in revolving the multiplicand-line through a right angle, either *anti-clockwise* or *clockwise.*

So, when the unit or operand is any vector whatever, ${}^{-}1$ has two distinct square roots, say i and i', whose effects as multipliers are to revolve the line through a right angle anti-clockwise and clockwise respectively. Hence the effect of i' as a multiplier is the same as if the multiplicand-line were first multiplied by i and then reversed, *i.e.*, were multiplied by $-i$; hence $i' = -i$, since both numbers give the same result when multiplying any same unit [I. § 1].

Since division is the inverse of multiplication, and consists in finding one factor when the product and the other factor are given [I. § 9], to divide a given vector by i is to find another vector that, if multiplied by i, would produce the given vector. The quotient is the vector got by revolving the dividend-vector through a right angle clockwise; for manifestly, when this quotient-vector is revolved through a right angle anti-clockwise, *i.e.*, is multiplied by i, the original direction is restored. Hence, to divide any vector by i is the same thing as to multiply it by $-i$; and, in like manner, to divide any vector by $-i$ is to revolve it through a right angle anti-clockwise, *i.e.*, to multiply it by i.

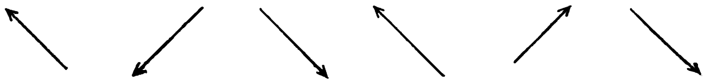
E.g., the unit $\longrightarrow$ gives the products and quotients:

unit $\times$ 1	unit $\times$ i	unit $\times$ ${}^{-}1$	unit $\times$ 1	unit $\times$ $-i$	unit $\times$ ${}^{-}1$
unit : 1	unit : $-i$	unit : ${}^{-}1$	unit : 1	unit : i	unit : ${}^{-}1$



So, the unit  gives the products and quotients:

unit $\times 1$	unit $\times\ i$	unit $\times\ ^{-}1$	unit $\times 1$	unit $\times -i$	unit $\times\ ^{-}1$
unit : 1	unit : $-i$	unit : $^{-}1$	unit : 1	unit : i	unit : $^{-}1$



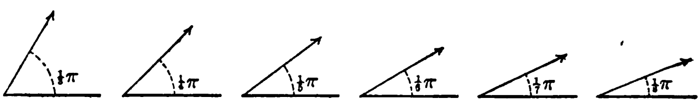
THE SYMBOLS $\sqrt[3]{^{-}1}$, $\sqrt[4]{^{-}1}$, ETC.

As the operation of multiplying by $^{-}1$ consists in reversing the quality of the multiplicand, and is represented by one reversal of the line that represents the multiplicand; and as the operation of multiplying by $\sqrt{^{-}1}$ is one which if twice performed reverses the quality of the multiplicand, and is represented by a half reversal of the line that represents the multiplicand:

So, multiplying by $\sqrt[3]{^{-}1}$ is an operation which three times performed reverses the multiplicand, and it is represented by one-third of one reversal of the line.

So, multiplying by $\sqrt[4]{^{-}1}$ is an operation which four times performed reverses the multiplicand, and it is represented by one-fourth of one reversal of the line; and so on.

The representatives of $\sqrt[3]{^{-}1}$, $\sqrt[4]{^{-}1}$, $\sqrt[5]{^{-}1}$, $\cdots$ are the rotations shown in the following figures, wherein lines of the same length as the unit make with that unit angles of $\frac{1}{3}\pi$, $\frac{1}{4}\pi$, $\frac{1}{5}\pi$, $\cdots$.



MULTIPLE ROOTS.

But $\because$ $(^{-}1)^1 = {^{-}1}$, $(^{-}1)^3 = {^{-}1}$, $(^{-}1)^5 = {^{-}1}$, $(^{-}1)^7 = {^{-}1}$, $\cdots$,

i.e., $\because$ 1, 3, 5, 7, $\cdots$ (any odd number) reversals has the same effect as one reversal,

$\therefore$ $\sqrt{^{-}1}$ may be represented by one-half of 1, 3, 5, 7, $\cdots$ reversals;

and $\because$ one-half of 5, 9, 13, $\cdots$ reversals are $2\frac{1}{2}$, $4\frac{1}{2}$, $6\frac{1}{2}$, $\cdots$ reversals, and have the same effect as a half reversal,

and one-half of 7, 11, 15, $\cdots$ reversals are $3\frac{1}{2}$, $5\frac{1}{2}$, $7\frac{1}{2}$, $\cdots$ reversals, and have the same effect as $1\frac{1}{2}$ reversals,

$\therefore$ $^{-}1$ has only two distinct square roots in this system.

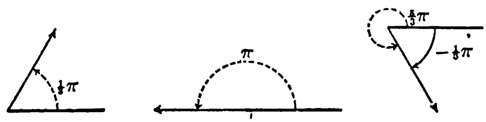
So, $\sqrt[3]{^{-}1}$ may be represented by $\frac{1}{3}$, $\frac{3}{3}$, $\frac{5}{3}$, $\frac{7}{3}$, $\cdots$ reversals;

and $\because$ $\frac{7}{3}, \frac{13}{3}, \frac{19}{3}, \cdots$ reversals $= 2\frac{1}{3}, 4\frac{1}{3}, 6\frac{1}{3}, \cdots$ rev'ls $= \frac{1}{3}$ rev'l,

and $\frac{9}{3}, \frac{15}{3}, \frac{21}{3}, \cdots$ reversals $= 3,\ 5,\ 7,\ \cdots$ rev'ls $= \frac{3}{3}$ rev'l,

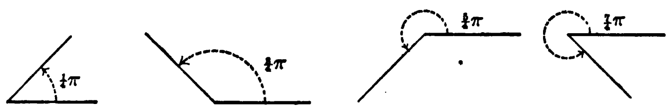
and $\frac{11}{3}, \frac{17}{3}, \cdots$ reversals $= 3\frac{2}{3}, 5\frac{2}{3}, 7\frac{2}{3}, \cdots$ rev'ls $= \frac{5}{3}$ rev'ls;

$\therefore$ $^{-}1$ has three cube roots represented by the curved arrow-lines of the figures, and but three.



So, $\sqrt[4]{^{-}1}$ may be represented by $\frac{1}{4}$, $\frac{3}{4}$, $\frac{5}{4}$, $\frac{7}{4}$, reversals,

and $^{-}1$ has four fourth roots represented by the arrow-lines of the figures, and but four; and so on.



POSITIVE AND NEGATIVE ROTATION.

Anti-clockwise rotation indicated by the figures is *positive rotation*, or rotation *through a positive angle;* and clockwise rotation is *negative rotation*, or rotation *through a negative angle.*

E.g., in the third figure above the two arrows indicate positive and negative rotation respectively: rotation through the positive angle $\frac{5}{3}\pi$ and through the negative angle $-\frac{1}{3}\pi$.

The roots of $^{-}1$ represented by negative rotation are therefore identical with those represented by positive rotation when taken in reverse order.

E.g., $(^{-}1)^{-\frac{1}{3}} = (^{-}1)^{\frac{5}{3}}$, $(^{-}1)^{-\frac{2}{3}} = (^{-}1)^{\frac{4}{3}}$, $(^{\pm}1)^{-\frac{3}{3}} = (^{\pm}1)^{\frac{3}{3}}$.

The reader may draw diagrams to illustrate.

MODULUS, ARGUMENT, VERSI-TENSOR.

Every number considered in algebra, whether real or imaginary, may be expressed in the form $r \cdot (^{-}1)^n$, wherein r is the tensor or quantitative factor of the number, and $(^{-}1)^n$ is the versor or qualitative factor. When the number $r(^{-}1)^n$ operates upon any vector, the result is a vector of like kind, such that r is the ratio of their lengths or magnitudes and n is the ratio $\theta : \pi$, which their difference of direction, θ, has to two right angles.

If n be an even number, $r(^{-}1)^n$ is positive; if odd, negative; if fractional, some or all of the values are imaginary.

The tensor r is also called the *modulus* of the number; $\theta, = n\pi$, is its *argument* or *versorial angle;* and the number $r \cdot (^{-}1)^n$ is a *versi-tensor.*

Every abstract number, whether real or imaginary, may be regarded as a versi-tensor.

E.g., $^{+}4$, $^{-}3$, $2i$, $-i$ are versi-tensors whose tensors are $^{+}4$, $^{+}3$, $^{+}2$, $^{+}1$, and versorial angles 0, π, $\frac{1}{2}\pi$, $\frac{3}{2}\pi$.

The reader should clearly distinguish between a vector and a tensor or versi-tensor. Vectors are lines, *i.e.*, quantities or concrete numbers, and may represent any concrete numbers, operands, or results, that admit of the same progressive change of quality as vectors undergo; but tensors and versi-tensors are abstract numbers, *i.e.*, ratios or operators, and are here represented by the relations of lines as to length and direction.

The product of any vector by a versi-tensor is a vector of like kind; that of two versi-tensors is a versi-tensor [§ 3].

The properties of versi-tensors are here explained and demonstrated by aid of the appropriate lines; but they would be as true, though perhaps not as evident, if standing alone in their symbolic form. It appears presently that versi-tensors are susceptible of all the ordinary operations of numbers when those operations are properly defined, and that the ordinary numbers of arithmetic and algebra are but special cases of these more general numbers. The same rules govern all sorts of numbers, and under these rules all sorts of numbers may be associated, and operated upon together without confusion or error.

§ 2. ADDITION AND SUBTRACTION.

In adding two or more numbers, two different results may be sought: (1) the arithmetic sum, or sum total, wherein no regard is paid to signs of quality; (2) the algebraic or net sum, wherein the quality and relations of the numbers are considered.

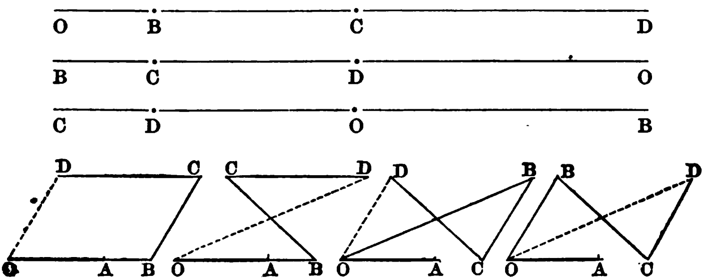
E.g., if a railway-train has run sixty miles east, and then forty miles west over the same track, the total mileage is one hundred miles; but the distance it now stands east of the starting point is but twenty miles.

So, if a sportsman walk ten miles east, then ten north and ten west, he walks thirty miles, but is only ten miles distant, and due north, from camp.

So, if several forces not all parallel to each other be applied to a body at the same point, the effective thrust, their resultant, is a single force acting along a line that may be parallel to neither of them and is less than their arithmetic sum.

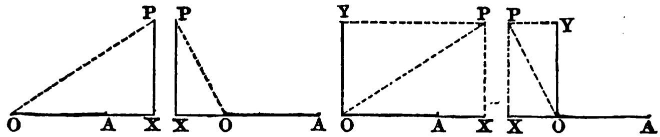
Two or more vectors are added by placing the initial point of the second upon the terminal point of the first, the initial point of the third upon the terminal point of the second, and so on, without changing their lengths or directions; and the *vector sum* is that line which joins the first initial to the last terminal point.

E.g., of the three lines OB, BC, CD, below, the vector sum is the line OD, whatever their length and direction; and this group of three lines, so far as the effect is concerned, in carrying the point from O to D, may be replaced by the single straight line OD.



In particular, the vector sum or difference of the two perpendiculars of a right triangle is the hypotenuse; and the vector sum of two adjacent sides of a rectangle is a diagonal.

E.g., in the figures below, OX + XP = OP and OX + OY = OP.



Conversely, a line may be replaced by any group of two or more lines that form a broken line and have the same initial and terminal points as the given line; and the diagonal of a parallelogram may be replaced by two adjacent sides.

E.g., in the figures above OD may be replaced by OB + BC + CD, and OP by OX + XP, or OX + OY.

The lines added are vectors (carriers), and their sum is a vector that reaches from the first initial to the last terminal point.

So, when abstract numbers, operators, are added together, viz., tensors, versors, and versi-tensors, their sum is a single operator that, acting upon a unit operand, produces the same result as if the several operators had acted separately upon the unit, and the results had then been added together. The sum of the several numbers is the same whatever vector be used as operand: for the vector sums got by using different operands, being obtained by like constructions, and so being homologous lines of similar figures, as also are the operands, bear like relations to the respective operands.

The *components* of a vector are any two perpendicular vectors of which it is the sum. A vertical vector has no horizontal component, and a horizontal vector has no vertical component. An operator that produces a vector perpendicular to the operand, or, more generally, that half reverses the quality of anything, is a *pure imaginary;* and an operator that produces an oblique vector is a *complex* imaginary.

E.g., in the right triangle OXP, let OA be the unit of length, and let OX, XP be respectively parallel and perpendicular to OA, and contain OA, in length a, b times;

then the symbols a, bi, $a+bi$, stand for numbers that acting as operators on the unit give the lines OX, XP, OP.

If the unit be horizontal, if r be the length of any vector, and θ be its inclination to the unit, then $r\cos\theta$ is the length of its horizontal component, and $r\sin\theta$ of its vertical component. The horizontal component is produced by an operator whose tensor is $r\cos\theta$ and whose versor is 1; the vertical component is produced by an operator whose tensor is $r\sin\theta$ and whose versor is i. Hence the oblique vector is produced by the operator $r(\cos\theta + i\sin\theta)$; and the operator $r(-1)^n$ $[n=\theta:\pi]$ is equivalent to the operator $r(\cos\theta+i\sin\theta)$. The first gives the number in its versi-tensorial form, as the product of a tensor and a versor; the second in its complex form, as the sum of its two *elements*; *i.e.*, of a real number and a pure imaginary.

If x, yi be the elements of any number, and r, θ the modulus and argument,

then $x=r\cos\theta$, $y=r\sin\theta$, $r=\sqrt[+]{(x^2+y^2)}$, $\theta=\tan^{-1}(y:x)$.

Any number $r(\cos\theta+i\sin\theta)$ is $\left\{\begin{matrix}\textit{larger}\\ \textit{smaller}\end{matrix}\right.$ than another number, when its modulus or tensor, r, is $\left\{\begin{matrix}\text{larger}\\ \text{smaller}\end{matrix}\right.$ than the modulus of the other: it is $\left\{\begin{matrix}\textit{greater}\\ \textit{less}\end{matrix}\right.$ than the other number when its *real element*, $r\cos\theta$, is $\left\{\begin{matrix}\text{greater}\\ \text{less}\end{matrix}\right.$ than the real element of the other. The relations expressed by the signs $\gtrdot$, $\lessdot$ are independent of quality or direction, and they depend only upon the lengths of the vectors produced, while the relations expressed by the signs $>$, $<$ depend only upon the horizontal projections.

A number is *infinitesimal*, *finite*, or *infinite* when its modulus is infinitesimal, finite, or infinite; and the arguments of 0 and ∞ are generally indeterminate.

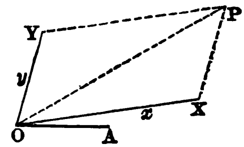
E.g., $-1\pm3i \gtrdot 1\pm i$, $-1\pm3i<1\pm i$, $0\cdot i=0$, $^\pm\infty\cdot i=\infty$.

If the modulus r and argument θ of a variable versi-tensor approach as limits the modulus r_1 and argument θ_1 of a finite constant versi-tensor, then $\cos\theta \doteq \cos\theta_1$, $\sin\theta\doteq\sin\theta_1$, and the elements $r\cos\theta$, $ri\sin\theta$ of the variable $\doteq$ the elements $r_1\cos\theta_1$, $r_1 i\sin\theta_1$ of the constant. Conversely, if the elements $r\cos\theta$,

$ri\sin\theta \doteq r_1\cos\theta_1$, $r_1 i\sin\theta_1$, then the quotient $r(i\sin\theta):r\cos\theta$, $=i\tan\theta$, $\doteq i\tan\theta_1$, and $\theta\doteq\theta_1$, and $r\doteq r_1$. The constant $r_1(\cos\theta_1+i\sin\theta_1)$ is then *the limit of the variable* $r(\cos\theta+i\sin\theta)$.

THEOR. 1. *Addition is commutative and associative.*

(*a*) *Two numbers.*



Let x, y be any two numbers; then will $x+y=y+x$.

For, let OA be any line, and let OX, OY be the results of operating upon this line by the numbers x, y; complete the parallelogram XOY-P;

then $\because$ OX = YP, OY = XP, both in magnitude and in direction, [geom., df. eq. lines

$\therefore$ OX + OY = OX + XP = OP; [df. add. lines

and OY + OX = OY + YP = OP,

$\therefore$ OX + OY = OY + OX, [II. ax. 1

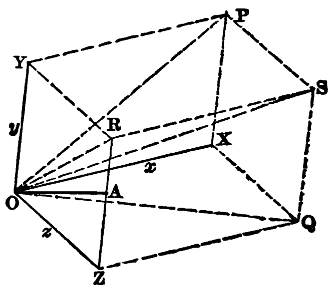
i.e., $x\cdot\mathrm{OA}+y\cdot\mathrm{OA}=y\cdot\mathrm{OA}+x\cdot\mathrm{OA}$,

$\therefore$ $x+y=y+x$. Q.E.D. [df. add. operators

(*b*) *Three numbers.*

Let x, y, z be any three numbers;

then will $\overline{x+y}+z=x+\overline{y+z}=\overline{z+y}+x=z+\overline{y+x}=\cdots$



For let OA be any line, and let OX, OY, OZ be the results of operating upon this line by the numbers x, y, z; complete the parallelograms XOY-P, XOZ-Q, YOZ-R, PXQ-S;

then $\because$ OX = YP = ZQ = RS,

OY = XP = ZR = QS,

OZ = XQ = YR = PS;

and $\because$ $\overline{\mathrm{OX}+\mathrm{OY}}+\mathrm{PS}=\mathrm{OS}$, $\mathrm{OX}+\overline{\mathrm{OY}+\mathrm{PS}}=\mathrm{OS}$, $\cdots$ [df. add. lines

$\therefore$ $\overline{\mathrm{OX}+\mathrm{OY}}+\mathrm{PS}=\mathrm{OX}+\overline{\mathrm{OY}+\mathrm{PS}}=\cdots$

i.e., $\overline{x+y}\cdot\mathrm{OA}+z\cdot\mathrm{OA}=x\cdot\mathrm{OA}+\overline{y+z}\cdot\mathrm{OA}$,

$\therefore$ $\overline{x+y}+z=x+\overline{y+z}=\cdots$ Q.E.D.

(*c*) *Four or more numbers.*

1. The theorem is true for two numbers, and for three. [(a, b)
2. If it be true up to n numbers inclusive, it is true also for $n+1$ numbers.

For let the $n+1$ numbers $x, y, z \cdots u, v$ be grouped and added together in any desired way, and let the sum be s;

then $\because$ s is obtained by adding the sum, say Q, of some of these numbers to the remaining number, or to the sum, say R, of the remaining numbers,

$\therefore$ $s = Q + R$.

Let R be that one of these sums which contains the number v,

and let T be the sum of the other numbers that make up R;

then $\because$ neither Q, R, T nor $Q+T$ contains more than n numbers,

$\therefore$ in each of them the several numbers may take any desired order and grouping; [hyp.

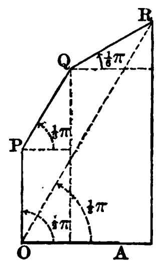
$$\therefore\ s, = Q + R, = Q + \overline{T + v}$$
$$= \overline{Q + T} + v \qquad [(b)$$
$$= \overline{x + y + z + \cdots + u} + v$$
$$= x + y + z + \cdots + u + v. \qquad \text{Q.E.D.}$$

3. But the theorem is true for three numbers; [(b)

$\therefore$ it is true for four numbers, [2

$\therefore$ for five numbers, for six numbers, and so on. Q.E.D.

Cor. 1. *The sum of two or more numbers is the sum of their elements.*



E.g., if the first root of each system be taken;

then
$$\sqrt[2]{-1} + \sqrt[3]{-1} + \sqrt[6]{-1}$$
$$= i + (\tfrac{1}{2} + \tfrac{1}{2} i\sqrt{3}) + (\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2} i)$$
$$= \tfrac{1}{2}[(1 + \sqrt{3}) + i(3 + \sqrt{3})]$$
$$= \tfrac{1}{2}(1 + \sqrt{3})(1 + i\sqrt{3})$$
$$= (1 + \sqrt{3})\sqrt[3]{-1}.$$

Cor. 2. *The modulus of the sum of two or more numbers is not greater than the sum of their moduli; and, if the imaginaries be unlike, it is less.*

Cor. 3. *If to the minuend the opposite of the subtrahend be added, the sum is the remainder sought.*

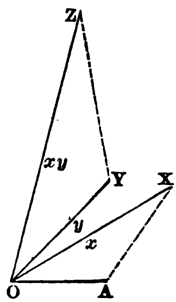
§ 3. MULTIPLICATION AND DIVISION.

To *multiply* a concrete quantity by a versi-tensor is to multiply the quantity by the tensor of the multiplier, and to make such reversal or partial reversal of the quality of the result as is shown by the versor of the multiplier.

E.g., let OX : OA be the multiplier, OY the multiplicand, OZ the product;

then as to magnitude, OZ : OY = OX : OA,

and as to quality, $\angle$YOZ = $\angle$AOX.



The *product* of two or more abstract numbers, versi-tensors, is a single number that operating as multiplier upon any unit produces at one operation the same result as if the several versi-tensors operated as multipliers in succession — the first upon the unit, the second upon the first product, taken as a new unit, and so on till all were used. The product of the several numbers is the same whatever vector be used as operand: for the vectors that result from using different operands are obtained from these operands by like constructions, and so bear like relations to the respective operands.

The *quotient* of one versi-tensor by another is that number which multiplied by the divisor gives the dividend as product. The dividend is likewise produced when the divisor is multiplied by the quotient: for, as appears presently [th. 3], the product is independent of the order of the factors.

Since, by definition [I. § 8], the product of two reciprocals is 1, it follows that the effect of their successive operation as multipliers upon any quantity is to leave the operand unchanged; and that to divide by a versi-tensor is to multiply by its reciprocal.

E.g., of the two square roots of $^-1$, viz., i and $-i$, the product is unity, and they are reciprocals: for [§ 1] their successive operation upon any vector leaves it unchanged, and to multiply by either is to divide by the other.

THEOR. 2. *If two or more numbers be multiplied together, the modulus of their product is the product of their moduli, and the argument of their product is the sum of their arguments.*

Let x, y, z, $\cdots$ be any numbers severally equal to

$$r\cdot({}^-1)^n,\quad r'\cdot({}^-1)^{n'},\quad r''\cdot({}^-1)^{n''},\quad \cdots,$$

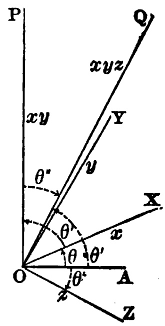
whose moduli are r, r', r'', $\cdots$, and whose arguments are

$$\theta,\ \theta',\ \theta'',\ \cdots,\ \text{equal to } n\pi,\ n'\pi,\ n''\pi,\ \cdots;$$

then will $x\cdot y\cdot z\cdot \cdots = r\cdot r'\cdot r''\cdots\cdot({}^-1)^{n+n'+n''+\cdots}$.

For let $\overline{OA}$ be the vector unit, and let x, y, z, $\cdots$, operating on the unit, produce the vectors $\overline{OX}$, $\overline{OY}$, $\overline{OZ}$, $\cdots$,

and let OA, OX, $\cdots \equiv$ the lengths of the vectors $\overline{OA}$, $\overline{OX}$, $\cdots$;



then $\because$ $\overline{OX} = r({}^-1)^n\cdot\overline{OA}$, $\overline{OP} = r'({}^-1)^{n'}\cdot\overline{OX}$,

$\therefore$ $\overline{OP} = r'({}^-1)^{n'}\cdot r({}^-1)^n\cdot\overline{OA}$;

but $\because$ OX $= r\cdot$ OA, and OP $= r'\cdot$ OX,

and $\because$ $\angle$AOX $= \theta$, and $\angle$XOP $= \theta'$,

$\therefore$ OP $= rr'\cdot$ OA, and $\angle$AOP $= \theta + \theta'$,

$\therefore$ $\overline{OP}$ may be produced by acting on $\overline{OA}$ with the single operator whose tensor is rr', and whose versorial angle is $\theta+\theta'$, or whose versor is $({}^-1)^{n+n'}$.

$\therefore$ $r'({}^-1)^{n'}\cdot r({}^-1)^n\cdot\overline{OA} = rr'({}^-1)^{n+n'}\cdot\overline{OA}$,

$\therefore$ $r'({}^-1)^{n'}\cdot r({}^-1)^n = rr'({}^-1)^{n+n'}$. Q.E.D. [df. product

So, if $\overline{OQ}$ be produced when the operator whose tensor is r'', = OZ : OA, and whose versorial angle is θ'', = AOZ, acts upon $\overline{OP}$;

then $\because$ $\overline{OQ}$ is also got when a single operator, with a tensor $rr'r''$, = OQ : OA, and an angle $\theta+\theta'+\theta''$, = AOQ, acts on $\overline{OA}$,

$\therefore$ $r''({}^-1)^{n''}\cdot r'({}^-1)^{n'}\cdot r({}^-1)^n\cdot\overline{OA} = rr'r''({}^-1)^{n+n'+n''}\cdot\overline{OA}$,

$\therefore$ $r''({}^-1)^{n''}\cdot r'({}^-1)^{n'}\cdot r({}^-1)^n = rr'r''({}^-1)^{n+n'+n''}$. Q.E.D.

COR. *If one number be divided by another, the modulus of the quotient is the quotient of the moduli, and the argument of the quotient is the argument of the dividend less that of the divisor.*

In particular, of a number and its reciprocal, the moduli are reciprocals and the arguments are opposites.

Theor. 3. *Multiplication is commutative and associative.*

Let x, y, z ⋯ be any numbers, severally equal to
$r\cdot({}^-1)^n,\ r'\cdot({}^-1)^{n'},\ r''\cdot({}^-1)^{n''},\cdots$;

then ∵ the product $\overline{x\cdot y}\cdot z\cdots=\overline{r\cdot r'}\cdot r''\cdots\cdot({}^-1)^{\overline{n+n'}+n''+\cdots}$,

and the product $x\cdot\overline{y\cdot z}\cdots=r\cdot\overline{r'\cdot r''}\cdots\cdot({}^-1)^{n+\overline{n'+n''}+\cdots}$,

and ∵ $\overline{r\cdot r'}\cdot r''\cdots=r\cdot\overline{r'\cdot r''}\cdots$,

and $\overline{n+n'}+n''+\cdots=n+\overline{n'+n''}+\cdots$;

and so for any other order or grouping, [II. ths. 1, 3

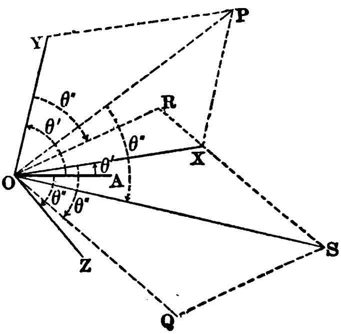
∴ the product $x\cdot y\cdot z\cdots$ is the same whatever the order and grouping of the factors. Q.E.D.

Theor. 4. *Multiplication is distributive as to addition.*

(*a*) *The product of the sum of two numbers by a third:*

Let x, y, z be any three numbers;

then will $z\cdot\overline{x+y}=z\cdot x+z\cdot y$.



Let $\overline{\text{OA}}$ be any vector unit, and let x, y, z, operating on the unit, produce $\overline{\text{OX}}$, $\overline{\text{OY}}$, $\overline{\text{OZ}}$, and let r, r', r''; θ, θ', θ'' be the moduli and arguments of x, y, z.

Complete the parallelogram XOY-P;

then $\overline{\text{OP}}=\overline{\text{OX}}+\overline{\text{OY}}$.

Turn $\overline{\text{OX}}$, $\overline{\text{OY}}$ by the angle θ'', and stretch them in the ratio r'', making $\overline{\text{OQ}}=z\cdot\overline{\text{OX}}$, $\overline{\text{OR}}=z\cdot\overline{\text{OY}}$.

Complete the parallelogram QOR-S;

then ∵ OQ : OX = OR : OY, and ∠XOQ = ∠YOR, [constr.

∴ ▱XOYP is similar to ▱QORS,

∴ ∠POS $=\theta''$ and $\overline{\text{OS}}=z\cdot\overline{\text{OP}}$;

and ∵ $\overline{\text{OS}}=\overline{\text{OQ}}+\overline{\text{OR}}$,

∴ $z\cdot\overline{\text{OP}}=z\cdot\overline{\text{OX}}+z\cdot\overline{\text{OY}}$; *i.e.*, $z\cdot\overline{\overline{\text{OX}}+\overline{\text{OY}}}=z\cdot\overline{\text{OX}}+z\cdot\overline{\text{OY}}$,

∴ $z\cdot\overline{x+y}\cdot\overline{\text{OA}}=z\cdot x\cdot\overline{\text{OA}}+z\cdot y\cdot\overline{\text{OA}}=\overline{z\cdot x+z\cdot y}\cdot\overline{\text{OA}}$,

∴ the product $z\cdot\overline{x+y}=$ the product $z\cdot x+z\cdot y$. Q.E.D.

(*b*) *The product of the sum of three or more numbers by another:*

Let x, y, z, $\cdots$ be three or more numbers, and v another;

then will $v\cdot\overline{x+y+z+\cdots}=v\cdot x+v\cdot y+v\cdot z+\cdots$.

$$\text{For}\quad v\cdot\overline{x+y+z+\cdots}=v\cdot x+v\cdot\overline{y+z+\cdots}\qquad[(a)$$
$$=v\cdot x+v\cdot y+v\cdot\overline{z+\cdots}$$
$$=v\cdot x+v\cdot y+v\cdot z+v\cdot\overline{\cdots};$$
$$=v\cdot x+v\cdot y+v\cdot z+\cdots.\quad\text{Q.E.D.}$$

(*c*) *The product of two or more polynomials:*

Let $x+y+z+\cdots$, $x'+y'+z'+\cdots$, be two polynomials;

$$\text{then}\quad \overline{x+y+z+\cdots}\cdot\overline{x'+y'+z'+\cdots}$$
$$=x\cdot\overline{x'+y'+z'+\cdots}$$
$$+y\cdot\overline{x'+y'+z'+\cdots}$$
$$+z\cdot\overline{x'+y'+z'+\cdots}+\cdots$$
$$=x\cdot x'+x\cdot y'+x\cdot z'+\cdots$$
$$+y\cdot x'+y\cdot y'+y\cdot z'+\cdots$$
$$+z\cdot x'+z\cdot y'+z\cdot z'+\cdots+\cdots.\quad\text{Q.E.D.}$$

So, if this product be multiplied by a third polynomial $\overline{x''+y''+z''+\cdots}$, a fourth, and so on.

COR. *The product of two or more complexes is the product of the sums of their elements used as polynomials.*

Let x, y be any complexes such that $x=p+qi$, $y=p'+q'i$; then will $x\cdot y=\overline{p+qi}\cdot\overline{p'+q'i}=pp'-qq'+i\,(pq'+p'q)$.

So for three or more factors.

NOTE. If x, y be put in the trigonometric form

$$r\cdot(\cos\theta+i\sin\theta),\quad r'\cdot(\cos\theta'+i\sin\theta'),$$

$$\text{then}\quad x\cdot y=rr'\cdot[(\cos\theta\cos\theta'-\sin\theta\sin\theta')$$
$$+i\,(\sin\theta\cos\theta'+\cos\theta\sin\theta')]$$
$$=rr'\cdot[\cos(\theta+\theta')+i\sin(\theta+\theta')].\qquad[\text{trig.}$$

So for three or more factors.

§ 4. POWERS AND ROOTS.

A $\left\{\begin{matrix}\text{positive}\\ \text{negative}\end{matrix}\right.$ *integral power*, $[r(^-1)^n]^{\pm m}$, of any versi-tensor $r(^-1)^n$, is the continued $\left\{\begin{matrix}\text{product.}\\ \text{quotient.}\end{matrix}\right.$

$$1 \underset{:}{\times} r(^-1)^n \underset{:}{\times} r(^-1)^n \cdots m \text{ times},$$

$$= 1 \underset{:}{\times} r^m \underset{:}{\times} [(^-1)^n]^m \qquad \text{[II. th. 3}$$

$$= 1 \times r^{\pm m} \times [(^-1)^n]^{\pm m}; \qquad \text{[df. int. pwr.}$$

i.e., it is a single versi-tensor that multiplying any vector quantity would at one operation stretch and turn it in the same way as if $r(^-1)^n$ had $\left\{\begin{matrix}\text{multiplied}\\ \text{divided}\end{matrix}\right.$ it m times in succession;

or, as if the tensor $r^{\pm m}$ had stretched it,

and as if the versor $[(^-1)^n]^{\pm m}$ had turned it m times as far as would the versor $(^-1)^n$, and in the $\left\{\begin{matrix}\text{same}\\ \text{opposite}\end{matrix}\right.$ direction.

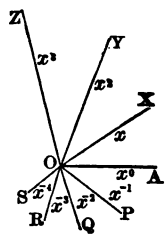
E.g., let OA be a unit, and let the ratio OX : OA be any imaginary x;

make $\angle$ AOX $= \angle$ XOY $= \angle$ YOZ $= \cdots$,

and OX : OA = OY : OX = OZ : OY = $\cdots$;

then the ratios OA : OA, OX : OA, OY : OA, $\cdots$, are the numbers x^0. x^1, x^2, $\cdots$,

and OA, OX, OY, $\cdots$, the results of using x as a multiplier $0, 1, 2, \cdots$ times upon OA.



So, the ratios OP : OA, OQ : OA, OR : OA, $\cdots$, are the numbers x^{-1}, x^{-2}, x^{-3}, $\cdots$,

and OP, OQ, OR, $\cdots$, are the results of using x as a divisor $1, 2, 3, \cdots$ times.

A *fractional power*, $[r(^-1)^n]^{\frac{p}{q}}$, is the pth power of any versi-tensor whose qth power is $r(^-1)^n$: *i.e.*, its effect when multiplying any vector quantity is to stretch the multiplicand in the ratio $r^{\frac{p}{q}}$, and to turn it as would the versor $[(^-1)^n]^{\frac{p}{q}}$, or $\frac{p}{q}$ times as far as would the versor $(^-1)^n$.

An *incommensurable power* $[r(^{-}1)^n]^m$ [m incommensurable], is the limit of $[r(^{-}1)^n]^{m'}$, wherein m' is a commensurable variable whose limit is m: *i.e.*, $[r(^{-}1)^n]^m$ denotes the versi-tensor whose $\left\{\begin{matrix}\text{modulus}\\\text{argument}\end{matrix}\right.$ is the limit of the $\left\{\begin{matrix}\text{modulus}\\\text{argument}\end{matrix}\right.$ of $[r(^{-}1)^n]^{m'}$ as $m' \doteq m$.

A $\left\{\begin{matrix}\textit{real}\\\textit{imaginary}\end{matrix}\right.$ *power* is any power whose exponent is $\left\{\begin{matrix}\text{real.}\\\text{imaginary.}\end{matrix}\right.$ The effect of an imaginary exponent is considered later.

THEOR. 5. *The modulus of any real power is the like power of the modulus of the base, and the argument of the power is the product of the argument of the base by the exponent of the power.*

Let $r(^{-}1)^n$ be any number whose modulus is r and argument $\theta, = n\pi$; and let m be any real exponent;

then will $[r(^{-}1)^n]^m = r^m(^{-}1)^{mn}$,

wherein r^m is the modulus of the product, and $m\theta, = mn\pi$, is its argument.

(*a*) m *a positive integer;*

then $[r(^{-}1)^n]^m = 1 \times r(^{-}1)^n r \times (^{-}1)^n \cdots m$ times [df. pos. int. pwr.

$= r \cdot r \cdots m \text{ times} \cdot (^{-}1)^{n+n \cdots m \text{ times}}$ [th. 2

$= r^m(^{-}1)^{mn}$. Q.E.D.

(*b*) m *a negative integer, say* $-$p;

then $\because [r(^{-}1)^n]^{-p} = 1 : [r(^{-}1)^n]^p$ [df. neg. int. pwr.

$= 1 : r^p(^{-}1)^{np}$ [(*a*)

$= r^{-p}(^{-}1)^{-np}$, [§ 3 th. 2 cr.

$\therefore [r(^{-}1)^n]^m = r^m(^{-}1)^{mn}$. Q.E.D.

(*c*) m *a positive or negative fraction* $\frac{\text{p}}{\text{q}}$; p, q *integers;*

then $\because [r^{\frac{1}{q}}(^{-}1)^{\frac{n}{q}}]^q = r(^{-}1)^n$, [(*a*)

$\therefore r^{\frac{1}{q}}(^{-}1)^{\frac{n}{q}} = [r(^{-}1)^n]^{\frac{1}{q}}$,

$\therefore r^{\frac{p}{q}}(^{-}1)^{\frac{p}{q}\cdot n} = [r(^{-}1)^n]^{\frac{p}{q}}$;

i.e., $[r(^{-}1)^n]^m = r^m(^{-}1)^{mn}$. Q.E.D.

(d) m *an incommensurable;*

Let m' be a commensurable exponent whose limit is m;

then $\because$ $r^m \equiv \lim$ tensor $r^{m'}$

and $\because$ $({}^-1)^{nm} \equiv \lim$ versor $({}^-1)^{nm'}$ [df. incom. pwr.

$\therefore$ $r^m({}^-1)^{nm} = \lim$ versi-tensor $r^{m'}({}^-1)^{nm'}$ [§ 2

$= \lim\,[r({}^-1)^n]^{m'}$ [a, b

$= [r({}^-1)^n]^m$. Q.E.D. [df. incom. pwr.

THEOR. 6. *Every finite number has* k *distinct* k*th roots, and no more, whose moduli are all equal and whose arguments are equidifferent.*

Let $r({}^-1)^n$ be any finite number, a the one real positive value of $\sqrt[k]{r}$, m any integer; [VIII. th. 13

then $\because$ $[a({}^-1)^{\frac{n+2m}{k}}]^k = r({}^-1)^{n+2m} = r({}^-1)^n$, [th. 5

$\therefore$ the several roots sought are:

$\cdots,\ a({}^-1)^{\frac{n-4}{k}},\ a({}^-1)^{\frac{n-2}{k}},\ a({}^-1)^{\frac{n}{k}},\ a({}^-1)^{\frac{n+2}{k}},\ a({}^-1)^{\frac{n+4}{k}},$

$\cdots$, all of the form $a({}^-1)^{\frac{n+2m}{k}}$,

wherein a is the modulus of all the roots alike,

and $\cdots,\ \dfrac{(n-4)\pi}{k},\ \dfrac{(n-2)\pi}{k},\ \dfrac{n\pi}{k},\ \dfrac{(n+2)\pi}{k},\ \dfrac{(n+4)\pi}{k},\ \cdots,$

are the arguments;

which differ from each other by a kth part of 2π.

But only k of these arguments have distinct effects, viz.:

$$\frac{n\pi}{k},\ \frac{(n+2)\pi}{k},\ \frac{(n+4)\pi}{k}\cdots\frac{(n+2\cdot\overline{k-1})\pi}{k};$$

for $\because$ the other arguments differ from those here named by entire revolutions,

$\therefore$ the corresponding roots are the products of these roots by even powers of ${}^-1$, and are identical with them.

And no number with modulus not a, or argument not embraced in the list above, can be a root.

For the kth power of any positive number not a is not r,

and the product of any other argument by k is not $(n+2m)\pi$.

NOTE 1. Of a positive real number the kth root takes the form $a(^-1)^{\frac{2m}{k}}$; of a negative real number, the form $a(^-1)^{\frac{1+2m}{k}}$.

NOTE 2. In the trigonometric form the theorem is written

$$\sqrt[k]{r(^-1)^n} = a\left[\cos\frac{(n+2m)\pi}{k} + i\sin\frac{(n+2m)\pi}{k}\right].$$

COR. 1. *Every finite number has* $^+$k *distinct* $\frac{\mathrm{h}}{\mathrm{k}}$*th powers and no more, their moduli all equal, and their arguments equidifferent.*

[h, k any integers prime to each other.

For $\because$ any $\frac{h}{k}$th power is the hth power of a kth root, [df. frac. pwr.

and $\because$ there are k such kth roots, whose common modulus is a, [above

and whose arguments all differ by multiples of $\frac{2\pi}{k}$ less than 2π,

say any two of them by $g\cdot\frac{2\pi}{k}$ [g any integer $\lessdot k$

$\therefore$ the corresponding arguments of the $\frac{h}{k}$th power differ by h times $g\cdot\frac{2\pi}{k}$, $=\frac{gh}{k}\cdot 2\pi$,

and $\because$ k does not measure gh, being $\gtrdot g$ and prime to h,

$\therefore$ this difference of arguments is not a multiple of 2π,

$\therefore$ all k of the $\frac{h}{k}$th powers are distinct in value;

and $\because$ their k arguments all differ by multiples of $\frac{2\pi}{k}$,

$\therefore$ when taken in order after rejecting all entire multiples of 2π, each differs from the next by $\frac{2\pi}{k}$;

i.e., the arguments of the powers are equidifferent. Q.E.D.

NOTE. The arguments are the same as for the kth root.

COR. 2. (a) *If a commensurable exponent* m′ *approach some limit* m, *whether commensurable or incommensurable, then every value of the power* $[\mathrm{r}(^-1)^{\mathrm{n}}]^{\mathrm{m}'}$ *approaches some value of* $[\mathrm{r}(^-1)^{\mathrm{n}}]^{\mathrm{m}}$ *as a limit.*

(*b*) *If* m *be incommensurable, the argument of the power* $[r(^-1)^n]^m$ *may be indeterminate.*

For $\because$ (*a*) the common modulus $^+r^{m'}$ of all values of $[r(^-1)^n]^{m'}$ approaches as a limit the modulus $^+r^m$ of $[r(^-1)^n]^m$,

and $\because$ the argument $m'(n+2k)\pi$ of any particular value of $[r(^-1)^n]^{m'}$ approaches as a limit the argument $m(n+2k)\pi$ of that corresponding value of $[r(^-1)^n]^m$ which is in the same series,

$\therefore$ every value of the power $[r(^-1)^n]^{m'}$ approaches some value of $[r(^-1)^n]^m$ as a limit. Q.E.D.

And $\because$ (*b*) as the commensurable m' approaches the incommensurable limit m, the successive convergents have larger and larger denominators, [continued fractions

$\therefore$ the number of distinct values of the m'th power increases without limit as $m' \doteq m$;

and $\because$ for any value of m' these numerous values of the power have their arguments equidifferent,

$\therefore$ as $m' \doteq m$ the arguments of consecutive values of the power approach one another more and more closely,

and in the limiting case, when the exponent is the incommensurable m, the argument of the power may be regarded as quite indeterminate, *i.e.*, as continuous. Q.E.D.

NOTE. By convention, however, the values of an incommensurable real power of a real positive base are often restricted to the single real positive value.

So, by convention, every power of the Napierian base e [XII. th. 28, ap. 4, cr.] is restricted to its real positive value, though the powers of the equivalent number 2.71828 ⋯ are not so restricted;

i.e., $\sqrt{e}=1{\cdot}64872\cdots$ only; but $\sqrt{2{\cdot}71828\cdots}=\pm 1.64872\cdots$.

THEOR. 7. *The product of like powers of two or more bases is the same power of the product of the bases.*

Let the bases be $r(^-1)^{\frac{\theta}{\pi}}$, $r'(^-1)^{\frac{\theta'}{\pi}}$ $\cdots$, whose moduli are $r, r', \cdots$, and whose arguments are $\theta, \theta', \cdots$; and let m be any real exponent:

then $\because$ of the powers $[r(^-1)^{\frac{\theta}{\pi}}]^m$, $[r'(^-1)^{\frac{\theta'}{\pi}}]^m, \cdots$,
the moduli are $r^m, r'^m, \cdots$,

and the arguments are $m\theta, m\theta', \cdots$;

$\therefore$ of the product of powers $[r(^-1)^{\frac{\theta}{\pi}}]^m \cdot [r'(^-1)^{\frac{\theta'}{\pi}}]^m \cdots$
the modulus is $r^m \cdot r'^m \cdots, = (rr'\cdots)^m$,

and the argument is $m\theta + m\theta' + \cdots, = m(\theta + \theta' + \cdots)$;

and $\because$ the product of the given bases has modulus $rr'\cdots$
and argument $\theta + \theta' + \cdots$,

$\therefore$ its mth power has modulus $(rr'\cdots)^m$
and argument $m(\theta + \theta' + \cdots)$;

i.e., the product of mth powers of the bases, and the mth power of the product of the bases, have the same modulus and argument, and are equal. Q.E.D.

COR. *The quotient of like powers of two bases is the same power of the quotient of the bases.*

NOTE. When the exponent m is commensurable, and the arguments $\theta, \theta', \cdots$ of the given bases are so related that the values of their sum $\theta + \theta' + \cdots$ cannot differ from one another except by certain of the multiples of 2π, it may happen that the power of the product or quotient has more distinct values than the product or quotient of the powers. [comp. VIII. th. 9

E.g., let two given bases, and their products, be $1 + i$, $2i$, $-2 + 2i$, whose moduli and arguments are:
$\sqrt{2}$, $\frac{1}{4}\pi + 2h\pi$; 2, $\frac{1}{2}\pi + 2k\pi$; $\sqrt[+]{8}$, $\frac{3}{4}\pi + 2l\pi$,

wherein $h, k, l \equiv$ any integers, positive, negative, or zero;

then, in general, $(1 + i)^m \cdot (2i)^m = (^-2 + 2i)^m$;

i.e., every value of either member is a value of the other member,

for the modulus of either member is $^+8^{\frac{m}{2}}$,

and the argument of either member is $m \cdot (\frac{3}{4}\pi + l \cdot 2\pi)$,

wherein $l, = h + k$, is any integer whatever.

But if $m=\frac{1}{3}$, and if it happen that in the investigation from which the bases $1+i$, $2i$, ^-2+2i arise, the $2i$ is got as the square of the $1+i$, while the ^-2+2i presents itself independently,

then $\because$ $k=2h$, while l remains unrestricted,

$\therefore$ the argument of $(1+i)^m \cdot (2i)^m$ is

$\frac{1}{3}(\frac{3}{4}\pi + h + 2h\cdot 2\pi), = \frac{1}{4}\pi + h\cdot 2\pi,$

while the argument of $(^-2+2i)^m$ is

$\frac{1}{3}(\frac{3}{4}\pi + l\cdot 2\pi), = \frac{1}{4}\pi + l\cdot\frac{2}{3}\pi;$

i.e., the product of the powers has only one value,

while the power of the product has three distinct values.

THEOR. 8. *The product of two powers of any same base, in any same series, is that power of the base whose exponent is the sum of their exponents, and is in the same series.*

NOTE. Different powers of a base are *in the same series*, when they arise from attributing to the base the same argument and not arguments differing by one or more entire revolutions; *i.e.*, when their bases are identical and not merely equivalent. [comp. VIII. th. 10

Let the base be A, $\equiv r(^-1)^{\frac{\theta}{\pi}}$, whose modulus and argument are r and θ; and let p, q, $\cdots$ be any real exponents;

then $\because$ A^p, A^q, $\cdots$ $A^{p+q+\cdots}$ have the moduli r^p, r^q, $\cdots$ $r^{p+q+\cdots}$

and the arguments $p\theta$, $q\theta$, $\cdots$ $(p+q+\cdots)\theta$, [th. 5

and $\because$ $r^p\cdot r^q \cdots = r^{p+q+\cdots}$, [VIII. th. 10

and $p\theta+q\theta+\cdots=(p+q+\cdots)\theta$,

$\therefore$ the product of the moduli of A^p, A^q, $\cdots$ is the modulus of $A^{p+q+\cdots}$,

and the sum of the arguments of A^p, A^q, $\cdots$ is the argument of $A^{p+q+\cdots}$;

i.e., the product $A^p\cdot A^q\cdots = A^{p+q+\cdots}$; Q.E.D. [th. 2

and $\because$ the argument of this product is $(p+q+\cdots)\theta$, and not $(p+q+\cdots+2k\pi)\theta$,

$\therefore$ the product is in the same series as the factors. Q.E.D.

COR. *The quotient of two powers of any same base, in any same series, is a power of the base whose exponent is the difference of the given exponents; and it is in the same series.*

THEOR. 9. *A power of a power of any base is that power of the base whose exponent is the product of the given exponents.*

Let the base be $A, = r(^-1)^{\frac{\theta}{\pi}}$; and let m, n be any exponents;

then $\because$ A^m has the modulus r^m and the argument $m\theta$,

$\therefore$ $(A^m)^n$ has the modulus $(r^m)^n$ and the argument $n(m\theta)$,

i.e., $(A^m)^n$ has the modulus r^{mn} and the argument $mn\theta$;

but A^{mn} has the same modulus r^{mn} and argument $mn\theta$;

$\therefore$ $(A^m)^n = A^{mn}$. Q.E.D.

NOTE. If a base B be not identical with A^m but only equivalent thereto, and if n have a denominator q; then B^n may have values not included among those of A^{mn}; and Theor. 9 may be stated as follows:

Of any number known merely to be equivalent to a given power of a given base, any given power includes among its values all values of that power of the given base whose exponent is the product of the given exponents. [comp. VIII. th. 11

E.g., if θ, the argument of A, be $a + 2h\pi$,

and if $B, = A^m$ but $\not\equiv A^m$, have argument $m\theta + 2k\pi$,

wherein h, k may take in succession all integral values,

then $\because$ $A^{m\frac{p}{q}}$ has the argument $\dfrac{mp}{q}\cdot a + \dfrac{hmp}{q}\cdot 2\pi$,

and $\because$ $B^{\frac{p}{q}}$ has the argument $\dfrac{mp}{q}\cdot a + \dfrac{hmp + kp}{q}\cdot 2\pi$,

$\therefore$ $B^{\frac{p}{q}}$ takes every value of $A^{m\frac{p}{q}}$, but it may be that $B^{\frac{p}{q}}$ takes other values besides.

PROB. 1. TO FIND THE nth ROOT OF ANY REAL NUMBER, $\pm a^n$:

Put x *for the roots sought; then:*

To find the nth *root of* a^n, *write* $x = a(^-1)^{\frac{0}{n}},\ a(^-1)^{\frac{2}{n}},$

$a(^-1)^{\frac{4}{n}},\ \cdots\ a(^-1)^{\frac{2n-2}{n}}$; [th. 5

i.e., *write* $x = a(\cos 0 + i \sin 0),\ a\left(\cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}\right), \cdots.$

To find the nth *root of* $-a^n$, *write* $x = a(^-1)^{\frac{1}{n}},\ a(^-1)^{\frac{3}{n}},$

$a(^-1)^{\frac{5}{n}},\ \cdots\ a(^-1)^{\frac{2n-1}{n}}$:

i.e., *write* $x = a\left(\cos\dfrac{\pi}{n} + i\sin\dfrac{\pi}{n}\right),\ a\left(\cos\dfrac{3\pi}{n} + i\sin\dfrac{3\pi}{n}\right), \cdots.$

1. To find the square root of a^2;

then $\because$ $x = a(\cos 0 + i\sin 0),\ a(\cos\frac{2}{2}\pi + i\sin\frac{2}{2}\pi),$

and $\because$ $\cos 0 = 1,\ \sin 0 = 0;\ \cos\pi = -1,\ \sin\pi = 0,$ [trig.

$\therefore\ x = a,\ -a.$

2. To find the square root of $-a^2$:

then $\because$ $x = a(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi),\ a(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi),$

and $\because$ $\cos\frac{1}{2}\pi = 0,\ \sin\frac{1}{2}\pi = 1;\ \cos\frac{3}{2}\pi = 0,\ \sin\frac{3}{2}\pi = -1,$ [trig.

$\therefore\ x = ai,\ -ai.$

3. To find the cube root of a^3:

then $\because$ $x = a(\cos 0 + i\sin 0),\ a(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi),$
$a(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi),$

and $\because$ $\cos 0 = 1,\ \sin 0 = 0;\ \cos\frac{2}{3}\pi = -\frac{1}{2},\ \sin\frac{2}{3}\pi = \frac{1}{2}\sqrt{3};$
$\cos\frac{4}{3}\pi = -\frac{1}{2},\ \sin\frac{4}{3}\pi = -\frac{1}{2}\sqrt{3},$ [trig.

$\therefore\ x = a,\ \frac{1}{2}a(-1 + i\sqrt{3}),\ \frac{1}{2}a(-1 - i\sqrt{3}).$

4. To find the cube root of $-a^3$:

then $\therefore$ $x = a(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi),\ a(\cos\frac{3}{3}\pi + i\sin\frac{3}{3}\pi),$
$a(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi),$

and $\because$ $\cos\frac{1}{3}\pi = \frac{1}{2},\ \sin\frac{1}{3}\pi = \frac{1}{2}\sqrt{3};\ \cos\pi = -1,\ \sin\pi = 0;$
$\cos\frac{5}{3}\pi = \frac{1}{2},\ \sin\frac{5}{3}\pi = -\frac{1}{2}\sqrt{3},$ [trig.

$\therefore\ x = \frac{1}{2}a(1 + i\sqrt{3}),\ -a,\ \frac{1}{2}a(1 - i\sqrt{3}).$

5. To find the fourth root of a^4:

then $\because$ $x = a(\cos 0 + i\sin 0),\ a(\cos\frac{2}{4}\pi + i\sin\frac{2}{4}\pi),$
$a(\cos\frac{4}{4}\pi + i\sin\frac{4}{4}\pi),\ a(\cos\frac{6}{4}\pi + i\sin\frac{6}{4}\pi),$

and $\because$ $\cos 0 = 1,\ \sin 0 = 0;\ \cos\frac{1}{2}\pi = 0,\ \sin\frac{1}{2}\pi = 1;$
$\cos\pi = -1, \sin\pi = 0;\ \cos\frac{3}{2}\pi = 0,\ \sin\frac{3}{2}\pi = -1;$ [trig.

$\therefore\ x = a,\ ai,\ -a,\ -ai.$

6. To find the fourth root of $-a^4$:

then $\because$ $x = a(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi),\ a(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi),$
$a(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi),\ a(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi),$

and $\because$ $\cos\frac{1}{4}\pi = \sqrt{\frac{1}{2}},\ \sin\frac{1}{4}\pi = \sqrt{\frac{1}{2}};\ \cos\frac{3}{4}\pi = -\sqrt{\frac{1}{2}},$
$\sin\frac{3}{4}\pi = \sqrt{\frac{1}{2}};\ \cos\frac{5}{4}\pi = -\sqrt{\frac{1}{2}},\ \sin\frac{5}{4}\pi = -\sqrt{\frac{1}{2}};$
$\cos\frac{7}{4}\pi = \sqrt{\frac{1}{2}},\ \sin\frac{7}{4} = -\sqrt{\frac{1}{2}};$ [trig.

$\therefore\ x = \frac{1}{2}a(\sqrt{2} + i\sqrt{2}),\ \frac{1}{2}a(-\sqrt{2} + i\sqrt{2}),$
$\frac{1}{2}a(-\sqrt{2} - i\sqrt{2}),\ \frac{1}{2}a(\sqrt{2} - i\sqrt{2}).$

7. To find the fifth root of a^5:

then $\because x = a(\cos 0 + i \sin 0),\ a(\cos \frac{2}{5}\pi + i \sin \frac{2}{5}\pi), \cdots,$

and $\because \cos 0 = 1,\ \sin 0 = 0;\ \cos \frac{2}{5}\pi = \frac{1}{4}(\sqrt{5} - 1),$

$\sin \frac{2}{5}\pi = \frac{1}{4}(10 + 2\sqrt{5});\cdots,$ [trig.

$$\therefore x = a,\ \tfrac{1}{4}a[(\sqrt{5} - 1) + i\sqrt{(10 + 2\sqrt{5})}],$$
$$\tfrac{1}{4}a[-(\sqrt{5} + 1) + i\sqrt{(10 - 2\sqrt{5})}],$$
$$\tfrac{1}{4}a[-(\sqrt{5} + 1) - i\sqrt{(10 - 2\sqrt{5})}],$$
$$\tfrac{1}{4}a[(\sqrt{5} - 1) - i\sqrt{(10 + 2\sqrt{5})}].$$

8. To find the fifth root of $-a^5$:

then $\because x = a(\cos \frac{1}{5}\pi + i \sin \frac{1}{5}\pi),\ a(\cos \frac{3}{5}\pi + i \sin \frac{3}{5}\pi), \cdots,$

and $\therefore \cos \frac{1}{5}\pi = \frac{1}{4}(\sqrt{5} + 1),\ \sin \frac{1}{5}\pi = \frac{1}{4}(10 - 2\sqrt{5});\cdots$ [trig.

$$\therefore x = \tfrac{1}{4}a[(\sqrt{5} + 1) + i\sqrt{(10 - 2\sqrt{5})}],$$
$$\tfrac{1}{4}a[-(\sqrt{5} - 1) + i\sqrt{(10 + 2\sqrt{5})}],\ -a,$$
$$\tfrac{1}{4}a[-(\sqrt{5} - 1 - i\sqrt{10 + 2\sqrt{5}})],$$
$$\tfrac{1}{4}a[(\sqrt{5} - 1) - i\sqrt{(10 - 2\sqrt{5})}].$$

And so for other roots.

PROB. 2. TO FIND THE nth ROOT OF AN IMAGINARY $a + bi$:

write $\mathrm{r} = \overset{+}{\sqrt{}}(\mathrm{a}^2 + \mathrm{b}^2)\ \theta = tan^{-1}(\mathrm{b} : \mathrm{a})$

then $\mathrm{a} + \mathrm{bi} = \mathrm{r}\,({}^-1)^{\frac{\theta}{\pi}},\ = \mathrm{r}(cos\,\theta + \mathrm{i}\ sin\,\theta);$

and $(\mathrm{a} + \mathrm{bi})^{\frac{1}{\mathrm{n}}} = \mathrm{r}^{\frac{1}{\mathrm{n}}}({}^-1)^{\frac{\theta}{\mathrm{n}\pi}},\ \mathrm{r}^{\frac{1}{\mathrm{n}}}({}^-1)^{\frac{\theta + 2\pi}{\mathrm{n}\pi}},\ \mathrm{r}^{\frac{1}{\mathrm{n}}}({}^-1)^{\frac{\theta + 4\pi}{\mathrm{n}\pi}},\ \cdots,$

$$= \mathrm{r}^{\frac{1}{\mathrm{n}}}\left(cos\,\frac{\theta}{\mathrm{n}} + \mathrm{i}\ sin\,\frac{\theta}{\mathrm{n}}\right),\ \mathrm{r}^{\frac{1}{\mathrm{n}}}\left(cos\,\frac{\theta + 2\pi}{\mathrm{n}} + \mathrm{i}\ sin\,\frac{\theta + 2\pi}{\mathrm{n}}\right), \cdots$$

E.g., to find the fourth root of $1 - \sqrt{{}^-3}$:

then $\because a = 1,\ b = -\sqrt{3},\ r = 2,\ \theta = \frac{2}{3}\pi,\ \frac{8}{3}\pi,\ \frac{14}{3}\pi,\ \frac{20}{3}\pi;$

$$\therefore 1 - \sqrt{{}^-3} = 2({}^-1)^{\frac{2}{3}},\ 2({}^-1)^{\frac{8}{3}},\ 2({}^-1)^{\frac{14}{3}},\ 2({}^-1)^{\frac{20}{3}},$$
$$= 2(\cos \tfrac{2}{3}\pi + i \sin \tfrac{2}{3}\pi),\quad 2(\cos \tfrac{8}{3}\pi + i \sin \tfrac{8}{3}\pi),$$
$$2(\cos \tfrac{14}{3}\pi + i \sin \tfrac{14}{3}\pi),\ 2(\cos \tfrac{20}{3}\pi + i \sin \tfrac{20}{3}\pi),$$
$$\therefore (1 - \sqrt{{}^-3})^{\frac{1}{4}} = 2^{\frac{1}{4}}({}^-1)^{\frac{1}{6}},\quad 2^{\frac{1}{4}}({}^-1)^{\frac{4}{6}},\ 2^{\frac{1}{4}}({}^-1)^{\frac{7}{6}},\ 2^{\frac{1}{4}}({}^-1)^{\frac{10}{6}}$$
$$= 2^{\frac{1}{4}}(\cos \tfrac{1}{6}\pi + i \sin \tfrac{1}{6}\pi),\quad 2^{\frac{1}{4}}(\cos \tfrac{4}{6}\pi + i \sin \tfrac{4}{6}\pi),$$
$$2^{\frac{1}{4}}(\cos \tfrac{7}{6}\pi + i \sin \tfrac{7}{6}\pi),\quad 2^{\frac{1}{4}}(\cos \tfrac{10}{6}\pi + i \sin \tfrac{10}{6}\pi),$$
$$= 1.0299 + .5945i,\quad -.5945 + 1.0299i,$$
$$-1.0299 - .5945i,\quad 5945 - 1.0299i.$$

§ 5. ABRIDGED REPRESENTATION.

In many important applications of the theory of imaginaries their representation is abridged as follows:

A fixed point or *origin* O is chosen, and a unit-line $\overline{\text{OA}}$ pointing to the right is taken as the common operand of all the imaginaries to be represented; then, of any imaginary $\overline{\text{OP}} : \overline{\text{OA}}$, $\overline{\text{OP}}$ is the *representative vector*, and P is the *representative point;* for, since the operand $\overline{\text{OA}}$ is the same for all the imaginaries, the resulting vectors or even their terminal points are sufficient to distinguish one operator from another.

In this abridged representation, the thing chiefly present to the mind is the point P; and every number, real or imaginary, is conceived to be written at its representative point, in the plane OAP.

E.g., if P, Q be the representative points of any numbers p, q, then P is further than Q $\left\{\begin{matrix}\text{to the right}\\ \text{from the origin}\end{matrix}\right.$ when $p \left\{\begin{matrix} > \\ \gtrless \end{matrix}\right. q$; the middle point of PQ is the representative point of $\frac{1}{2}(p+q)$; and $\overline{\text{PQ}}$ is the representative vector of $(q-p)$.

So, if A, B, C, D be the representative points of a, b, c, d respectively, and if $a+c=b+d$, then ABCD is a parallelogram whose equal sides AB, DC are representative vectors of the equal numbers $(b-a)$, $(c-d)$, and whose centre is the representative point of $\frac{1}{4}(a+b+c+d)$.

If a variable pass from one value to another by continuous change, then its representative point moves along some locus, the *path* of the variable.

E.g., the path of a real variable lies in the line OA; the path of a variable pure imaginary lies in the line through O perpendicular to OA; the path of a variable whose versorial angle is constant is a straight line through O; the path of a variable whose tensor is constant is the circumference of a circle whose centre is O.

§ 6. EXAMPLES.

§ 1.

1. Assume any convenient linear unit, and plat the numbers:
$0(^-1)^0$; $1(^-1)^1, -1(^-1)^{-1}$; $2(^-1)^2, -2(^-1)^{-2}$;
$3(^-1)^3, -3(^-1)^{-3}$; $4(^-1)^4, -4(^-1)^{-4}$; $5(^-1)^5, -5(^-1)^{-5}$.

2. So, the two values of $\sqrt{4}$ and of $\sqrt{^-4}$; the three values of $\sqrt[3]{27}$, and of $\sqrt[3]{^-27}$; the four values of $\sqrt[4]{256}$, and of $\sqrt[4]{^-256}$; the five values of $\sqrt[5]{3125}$, and of $\sqrt[5]{^-3125}$; the six values of $\sqrt[6]{216}$, and of $\sqrt[6]{^-216}$; the eight values of $\sqrt[8]{a^8}$ and of $\sqrt[8]{^-a^8}$; the ten values of $\sqrt[10]{a^{10}}$ and of $\sqrt[10]{^-a^{10}}$; the twelve values of $\sqrt[12]{a^{12}}$ and of $\sqrt[12]{^-a^{12}}$, wherein a is any positive real number.

§ 2.

3. Find the moduli and arguments of the following numbers, given by their elements:
$\frac{1}{2}+\frac{1}{2}\sqrt{^-3}$, $\frac{1}{2}-\frac{1}{2}\sqrt{^-3}$; $1+i$, $1-i$; $2\sqrt{3}+2i$, $2\sqrt{3}-2i$;
$\sqrt{5}+1 \pm i(10-2\sqrt{5})$, $-\sqrt{5}-1 \pm i(10-2\sqrt{5})$,
wherein $\frac{1}{4}(\sqrt{5}+1) = \cos 36°$, $\frac{1}{4}(10-2\sqrt{5}) = \sin 36°$;
$\sqrt{5}-1 \pm i(10+2\sqrt{5})$, $-\sqrt{5}+1 \pm i(10+2\sqrt{5})$,
wherein $\frac{1}{4}(\sqrt{5}-1) = \cos 72°$, $\frac{1}{4}(10+2\sqrt{5}) = \sin 72°$.

4. If a, b be any two real numbers, show that the modulus of the complex imaginary $a \pm bi$ is $\sqrt{(a^2+b^2)}$, and its argument, $\tan^{-1}(\pm b : a)$.

5. Assume any convenient linear unit, and plat the following numbers; add them, compute the moduli and arguments of the several sums, and plat those sums.
$2i^{\frac{4}{3}}, 2i^{\frac{8}{3}}$; $2i^2, 2i^{\frac{2}{3}}, 2i^{-\frac{2}{3}}$; $3i^{\frac{2}{3}}, 3i^{\frac{4}{3}}, 3i^2$; $3i^{-2}, 3i^{-\frac{8}{3}}, 3i^{-\frac{10}{3}}$.

6. Draw an equilateral triangle ABC, assume the base AB as the linear unit, then show what numbers will produce the sides AB, BC, CA, and find their elements.
Add these numbers and show that their sum is 0.

7. So, for the square, the regular pentagon, the regular hexagon, the regular octagon, the regular decagon, the regular dodecagon, assuming any side as unit.

§ 3.

8. Find the products, and show graphically that they are real, of $2+3i,\ 2-3i;\ a+bi,\ a-bi;\ \sqrt{3}+i\sqrt{5},\ \sqrt{3}-i\sqrt{5};\ \sqrt{2}+\sqrt{3}+i\sqrt{5},\ \sqrt{2}-\sqrt{3}+i\sqrt{5},\ 3+i\sqrt{10}$.

9. Find the products of the sets of numbers in Exs. 5, 6.

10. Multiply $a+bi$ by $c+di$, and show that the product takes the form $P+Qi$, wherein $P=ac-bd$, $Q=bc+ad$; and that $r=\sqrt{(a^2+b^2)(c^2+d^2)}$, and $\theta=\tan^{-1}\dfrac{bc+ad}{ac-bd}$.

11. Divide $a+bi$ by $c+di$, and show that the quotient takes the form $P+Qi$, wherein $P=\dfrac{ac+bd}{c^2+d^2}$, $Q=\dfrac{bc-ad}{c^2+d^2}$; and that $r=\sqrt{\dfrac{a^2+b^2}{c^2+d^2}}$, and $\theta=\tan^{-1}\dfrac{bc-ad}{ac+bd}$.

12. Divide $\dfrac{a+bi}{a-bi}:\dfrac{a-bi}{a+bi};\ \dfrac{a+bi}{c+di}:\dfrac{a-bi}{c-di};\ \dfrac{a-bi}{c+di}:\dfrac{a+bi}{c-di}$.

13. Multiply

$$2i^{\frac{4}{3}}+2i^{\frac{8}{3}} \text{ by } 2i^2+2i^{\frac{2}{3}};\ 3i^{\frac{2}{3}}+3i^{\frac{4}{3}}+3i^2 \text{ by } 3i^{-2}+3i^{-\frac{8}{3}}+3i^{-\frac{10}{3}}.$$

Plat the products; find their elements, moduli, and arguments.

14. Express two or more versi-tensors as complexes, and using them in that form, show that the multiplication of such complexes is both commutative and associative.

§ 4.

15. Resolve the numbers given in Ex. 2 into their elements.

16. Find the product of the two values of $\sqrt{4}$, and of $\sqrt{{}^-4}$;
of the three values of $\sqrt[3]{27}$, and of $\sqrt[3]{{}^-27}$;
of the four values of $\sqrt[4]{256}$, and of $\sqrt[4]{{}^-256}$;
of the five values of $\sqrt[5]{3125}$, and of $\sqrt[5]{{}^-3125}$.

Write the several factors in the three forms $r({}^-1)^n$, $r\cdot i^{2n}$, and $a+bi$; and show that the products so found are the same whichever form be used.

17. Find the powers and roots as indicated:

$$(i^0+i^2)^2;\ (i+i^2)^3;\ (2+2i\sqrt{3})^4;\ (i^0+i^{\frac{1}{3}}+i^{\frac{2}{3}})^3;$$
$$[2+\tfrac{1}{2}(\sqrt{5}+1)+\tfrac{1}{2}i(10-2\sqrt{5})]^4;\ (3i+3i^2)^{\frac{1}{3}}.$$

XI. EQUATIONS.

☞ For definition of the words *equation*, *identity*, *inequality*, *statement*, *member*, and *copula*, see I. § 5.

§ 1. STATEMENTS.

One statement is a $\left\{\begin{matrix}\textit{necessary}\\ \textit{sufficient}\end{matrix}\right.$ *condition* of another when, if the first be $\left\{\begin{matrix}\text{false,}\\ \text{true,}\end{matrix}\right.$ the other is also $\left\{\begin{matrix}\text{false.}\\ \text{true.}\end{matrix}\right.$

Two statements are *equivalent* if one be both a necessary and a sufficient condition of the other, *i.e.*, if they be false together and true together.

Two or more statements are $\left\{\begin{matrix}\textit{associated}\\ \textit{incompatible}\end{matrix}\right.$ when, if some of them be true, the others must be $\left\{\begin{matrix}\text{true ;}\\ \text{false ;}\end{matrix}\right.$ *independent* when, whichever of them be true or false, the rest may, just as well, be true, or be false.

There are n $\left\{\begin{matrix}\textit{necessary conditions}\\ \textit{contradictions}\end{matrix}\right.$ among m $\left\{\begin{matrix}\text{associated}\\ \text{incompatible}\end{matrix}\right.$ statements when some $m-n$ of these statements are independent, and if these be true, the remaining statements are $\left\{\begin{matrix}\text{all true.}\\ \text{all false.}\end{matrix}\right.$

E.g., the equation $x=3$ is equivalent to the equation $2x=6$, and it is a sufficient, but not a necessary condition, of the inequality $x<4$.

The three statements are associated and have two necessary conditions among them, since, if the first be true, so are both the others.

The last two are necessary conditions of the first; the third is not, but the second is, a sufficient condition of the first.

So, the inequalities $x<y$, $y<z$, $z<x$ are incompatible, involving one contradiction; for, though any two of the statements may be true or false, or one be true and the other false, yet if two of them be true, the third must be false.

So, the equations $x+y=2$, $x+2y=3$, $x+3y=\left\{\begin{matrix}4\\ 5\end{matrix}\right.$ are $\left\{\begin{matrix}\text{associates,}\\ \text{incompatibles,}\end{matrix}\right.$ having one $\left\{\begin{matrix}\text{necessary condition}\\ \text{contradiction}\end{matrix}\right.$ among them.

§ 2. SOLUTION OF EQUATIONS. — UNKNOWNS.

The letter or letters for which particular values are sought that shall make true the statements contained in the equations are the *unknown*, as distinguished from the other elements, that are given and now called *known* elements; and the *solution* of an equation, or system of equations, consists in making such transformations therein, as, while the equality of the members is preserved, and the relations between the elements are unchanged, shall result in giving the values of the unknown elements in terms of the known elements. The values so found are the *roots* of the equation or system of equations; and the test to be applied to them is to replace the unknown elements by these values, and see if they make the equations identities.

E.g., of the equation $2x = 4$ [x unknown] 2 is a root,

$\because\ 2 \cdot 2 \equiv 4$, a numerical identity. [df. root

So, of the equation $x^2 - 5x + 6 = 0$ [x unknown] 2, 3 are roots,

$\because\ 2^2 - 5 \cdot 2 + 6 \equiv 0,\ \ 3^2 - 5 \cdot 3 + 6 \equiv 0.$

So, of the equation $x^3 = a^3$ [x unknown],

$a,\ \frac{1}{2}a(-1 + i\sqrt{3}),\ \frac{1}{2}a(-1 - i\sqrt{3})$ are roots,

$\because\ a^3 \equiv a^3,\ [\frac{1}{2}a(-1 + i\sqrt{3})]^3 \equiv a^3,\ [\frac{1}{2}a(-1 - i\sqrt{3})]^3 \equiv a^3.$

Equations that involve the same unknown elements, and are satisfied by the same values of them, are *simultaneous* equations; and those values are *simultaneous values.*

E.g., if the equations $2x + 5y = 19$, $6x - 3y = 3$ [x, y unknown] be simultaneous, 2, 3 is a pair of roots,

$\because\ 2 \cdot 2 + 5 \cdot 3 = 19$, and $6 \cdot 2 - 3 \cdot 3 = 3.$

So, of the simultaneous equations $x - y = 5$, $x^2 + y^2 = 13$ $2, {}^-3$; $3, {}^-2$ are pairs of roots; but not $2, {}^-2$; $3, {}^-3$.

So, if two plane curves be expressed by two equations involving two variables, for the points of meeting both curves have the same coördinates, and for these points, but for no others, the two equations are simultaneous.

The roots of an equation are sometimes called its solution.

§ 3. DEGREE OF EQUATION.

If, without extracting roots, an equation involving one unknown element be so transformed that both members are entire as to that element, the *degree* of the equation is the degree of that term wherein the degree of the unknown element is highest. If the equation contain every power of the unknown element, from the highest to the zero power inclusive, it is a *complete* equation; if not, it is *incomplete*.

E.g., the equation $abx = cd + ef$ [x unknown] is of the first degree, a *simple equation*.

So, the equation $y^2 + 3y = 49$ [y unknown] is of the second degree; it is a *complete quadratic equation;*

but the equation $y^2 = 49$ is an *incomplete* quadratic;

and the equation $y^2 + 0y = 49$ is a complete quadratic.

So, the equation $r^3 + 5r^2 + 5r = 426$ [r unknown] is of the third degree; it is a *complete cubic equation.*

So, the equation $k^4 + 12k^3 + 60k^2 + 95k \equiv 1230$ [k unknown] is of the fourth degree; it is a *complete biquadratic equation.*

So, the equation $s = a(r^n - 1):(r - 1)$ is of the nth degree if r be the unknown element;
of the first degree, if s or a be the unknown element;
an *exponential equation* if n be the unknown element.

An equation may contain one unknown element or more.

E.g., the equations above have each one unknown element;

but the equation $ax^2 + 2hy + by^2 + 2gx + 2fy + d = 0$ is a complete quadratic with two unknown elements;

and $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz + d = 0$ [x, y, z unknown] is a complete quadratic involving three unknown elements.

If an entire equation involve two or more unknown elements, the degree of the equation is the sum of the exponents of the elements in that term in which their sum is greatest.

E.g., the equation $3xy^2 + 2x^2 + y^2 + x + y + 27 = 0$ is a cubic, but not complete.

§ 4. GENERAL PROPERTIES.

THEOR. 1. *If to both members of an equation the same number be added, the roots of the equation are not changed thereby.*

Let $P = Q$ be any equation, and N any number;
then are the roots of equations $P = Q$, $P + N = Q + N$, identical.

For $\because$ $P + N = Q + N$ when $P = Q$, and then only,

$\therefore$ every root or set of roots of the equation $P = Q$, satisfies the equation $P + N = Q + N$, and conversely;

i.e., every root of either equation is a root of the other.

Q.E.D.

COR. 1. *If any term be transposed from one side of an equation to the other and its sign reversed the roots of the equation are not changed thereby.*

E.g., the roots of the equations

$ax^2 + bx + c = 0$, $ax^2 + bx = -c$, are identical.

COR. 2. *If the signs of all the terms of an equation be changed the roots of the equation are not changed thereby.*

THEOR. 2. *If both members of an equation be multiplied by any same number, not a function of the unknown elements and not 0 nor ∞, the roots of the equation are not changed thereby.*

Let the equation $P = Q$ be any equation, and N any number not a function of the unknown elements, and not 0 nor ∞;

then are the roots of the equations $P = Q$, $N \cdot P = N \cdot Q$ identical.

For, write the equations in the form $P - Q = 0$, $N(P - Q) = 0$;

then $\because$ N is not a function of the unknown elements, and not 0, nor ∞,

$\therefore$ $N(P - Q)$ vanishes when $P - Q$ vanishes, and then only, and conversely;

$\therefore$ every root of either equation is a root of the other.

Q.E.D.

NOTE 1. If N be 0, $N(P - Q)$ vanishes for any values of the unknown elements that make $P - Q$ finite.

If N be ∞, $N(P - Q)$ may not vanish when $P - Q$ vanishes.

If N be a function of the unknown elements, N may vanish for other values of those elements than those values that make $P - Q$ vanish, and may thus cause $N(P - Q)$ to vanish.

If N be a function of the unknown elements, N may become infinite for some of the values that make $P - Q$ vanish, and $N(P - Q)$ may not vanish.

In each of these four cases the equations $P - Q = 0$, $N(P - Q) = 0$ may not have all their roots identical.

E.g., let X be any entire function of x, and a any constant;
then ∵ whatever factors X has, the product $(x - a) \cdot X$ has another factor, $x - a$.

∴ whatever roots the equation $X = 0$ has, the equation $(x - a) = 0 \cdot X$ has also the root a.

So, of the equation $x^2 - 5x + 6 = 0$, the roots are 2, 3;
but of the equation $x^3 - 5x^2 + 6x = 0$, the roots are 0, 2, 3,
i.e., by multiplying the equation by x a new root, 0, is introduced which does not satisfy the original equation $x^2 - 5x + 6 = 0$, and is not a root of it.

So, of the equation $x^3 - 5x^2 + 6x = 0$ the roots are 0, 2, 3;
but of the equation $x^2 - 5x + 6 = 0$ the roots are 2, 3, only,
i.e., by dividing the equation by x, one root, 0, is lost.

So, of the equation $3 - x = 15 - 2x$, the single root is 12;
but if this equation be multiplied by $x - 1$, the resulting equation $x^2 - 13x + 12 = 0$, has two roots, 12, 1.

So, of the equation $x^2 - 1 = ax - a$ the roots are $1, a - 1$;
and if both members be multiplied by $x : (x - 1)$, the resulting equation, $x(x + 1) = ax$, still has the root $x = a - 1$, for which the multiplier becomes neither ∞ nor 0,

but it has $\left\{ \begin{matrix} \text{lost } 1 \\ \text{gained } 0 \end{matrix} \right.$ for which the multiplier becomes $\left\{ \begin{matrix} \infty. \\ 0. \end{matrix} \right.$

NOTE 2. If the function X be not entire, but contain a term of the form $A : (x - a)$; then X may become infinite, when $x = a$, and $(x - a) \cdot X$ may take the form $0 \cdot \infty$, which may or may not vanish; and multiplying the equation $X = 0$ by $x - a$ may or may not introduce a new root into the equation.

E.g., multiplication by the factors x, $x+1$ generally introduces the roots $0, ^-1$;

but if the equation $2 = \frac{1}{x} + x - 1 + \frac{3}{x+1}$ be multiplied by $x \cdot (x+1)$, $0, ^-1$ are not introduced as new roots.

For $\because$ x contains a term $\frac{3}{x+1}$ which is infinite when $x = -1$,

and $\because$ this infinite term, when multiplied by the zero, $x+1$, is the finite number 3,

$\therefore$ $(x+1)\cdot$x does not vanish when $x = {}^-1$; and $^-1$ is not a root of the new equation.

So, $x \cdot$x does not vanish when $x = 0$; and 0 is not a root.

But if the equation $1 - \frac{x^2}{x-1} = \frac{1}{1-x} - 6$ be multiplied by $x-1$, the resulting equation is $x^2 - 7x + 6 = 0$, whose roots are 6, 1,

whereof 6 satisfies the original equation and is a root of it;

but $\because$ 1 does not satisfy it, and is not a root of it,

$\therefore$ by the use of the factor $x - 1$ a new root (a stranger) has been introduced into the equation.

The reason is manifest: the factor $x-1$ is not needed to clear the equation of fractions; for if the terms of the original equation be all transposed to one side and reduced to lowest terms, the equation becomes $7 - \frac{1-x^2}{1-x} = 0$,

i.e., $7-(1+x)=0$, whence $x=6$; and there is no other root;

i.e., the numerator and denominator vanish together when $x=1$, and the value of the fraction $0:0$ is 2.

So, the equation $1 + \frac{1}{6} - x + \frac{x^3-1}{x^2-1} = 0$ may be cleared of fractions by multiplying by $6(x-1)(x+1)$, and becomes $7x^2 + 6x - 13 = 0$, whose roots are 1, $-\frac{13}{7}$;

but $\because$ $-\frac{13}{7}$ satisfies the original equation, and 1 does not,

$\therefore$ the factor $x-1$ introduces a new root 1, but $x+1$ does not introduce a new root.

The reader may search out the reason for this difference.

So, if the equation $\dfrac{1}{x-a}+\dfrac{1}{x+a}-\dfrac{1}{x^2-a^2}=0$ be multiplied by all its denominators, the resulting equation is $(2x-1)(x^2-a^2)=0$, whose roots are $\frac{1}{2}, +a, -a$;

but if it be multiplied by the least common denominator, the resulting equation, $2x-1=0$, has a single root, $\frac{1}{2}$.

Of these three roots only $\frac{1}{2}$ satisfies the original equation.

THEOR. 3. *If the two members of an equation be raised to the same integral power, the results are equal; but it is possible that the new equation may have some roots not found in the old one.*

For if $P=Q$, wherein P or Q or both of them are functions of some unknown element, say x,

then $P^2=Q^2,\ P^3=Q^3,\ \cdots,\ P^n=Q^n,$ [II. ax. 6

$P^2-Q^2=0,\ P^3-Q^3=0,\ \cdots,\ P^n-Q^n=0,$

i.e., $(P-Q)(P+Q)=0,\ (P-Q)(P^2+PQ+Q^2)=0,\ \cdots,$

$(P-Q)(P^{n-1}+P^{n-2}Q\cdots+Q^{n-1})=0.$

But these equations are satisfied either if such values be given the unknown that $P-Q=0$,

or that $P+Q=0,\ P^2+PQ+Q^2=0,\cdots P^{n-1}+P^{n-2}Q\cdots+Q^{n-1}=0;$

and in general the roots of the equation $P-Q=0$ are not the same as the roots of the equations $P+Q=0$, $P^2+PQ+Q^2=0,\ \cdots,\ P^{n-1}+P^{n-2}Q+\cdots+Q^{n-1}=0.$

E.g., if $x=5$, then $x^2=25$, and $x={}^{+}5, {}^{-}5$;

but only ${}^{+}5$ satisfies the original equation and is its root.

So, if $\overset{+}{\sqrt{}}(9-x)=x-9$, then $x^2-17x+72=0$, and $x=8, 9$;

but 9, not 8, satisfies the original equation, and is its root.

Were that equation $\overset{-}{\sqrt{}}(9-x)=x-9$, the root were 8, not 9.

NOTE. Unless the reader be sure that every step he has taken is $\left\{\begin{matrix}\textit{valid,}\\ \textit{reversible,}\end{matrix}\right.$ *i.e.*, that each successive transformed equation is true $\left\{\begin{matrix}\text{whenever}\\ \text{only when}\end{matrix}\right.$ the previous ones are true, his results can serve merely to suggest values of the roots for trial. If any step has been $\left\{\begin{matrix}\text{invalid}\\ \text{irreversible}\end{matrix}\right.$ the problem may have $\left\{\begin{matrix}\text{fewer}\\ \text{other}\end{matrix}\right.$ solu-

tions than he has found. In particular, he must have multiplied by no more factors containing the unknown than were necessary to clear of fractions, and must have $\left\{\begin{matrix}\text{gained}\\\text{lost}\end{matrix}\right.$ no solutions in taking like $\left\{\begin{matrix}\text{powers}\\\text{roots}\end{matrix}\right.$ of both members; or else he must test his results by substituting in the original equation or equations, and say: if they satisfy the equation they are among its true roots; if not, they are strangers introduced in course of the work. The results are to be trusted only after they are tested.

THEOR. 4. *If all the terms of a rational integral equation involving one unknown element be transposed to one side, then:*

1. *The polynomial so formed has for a factor the excess of the unknown element over any root of the equation;*

2. *Conversely, if this polynomial have for a factor the excess of the unknown element over any given number, that number is a root of the equation.*

1. Let the equation $\text{X} = 0$ be any equation wherein X stands for some rational integral function of an unknown element x, say $\text{A}x^n + \text{B}x^{n-1} + \text{C}x^{n-2} + \cdots + \text{R}x^2 + \text{S}x + \text{T}$,

and let a be a root of the equation $\text{X} = 0$,

then is X measured by $x - a$.

For, divide X by $x - a$, and put Q, R for quotient and remainder;

then $\because$ $\text{X} = \text{Q}\cdot(x-a) + \text{R}$, for every value of x,

wherein R is independent of x and constant;

and $\because$ $\text{X} = 0$ and $x - a = 0$ when $x = a$, [hyp.

$\therefore$ $\text{R} = 0$;

and $\because$ the division of X by $x - a$ is effected without remainder,

$\therefore$ $x - a$ is a measure of X. Q.E.D.

2. Let X be any rational integral function of x, and let $x - a$ be a measure of X;

then is a a root of the equation $\text{X} = 0$.

For $\because$ $\text{X} = \text{Q}\cdot(x-a)$, for every value of x, and there is no remainder, [hyp.

and $\because$ $x - a = 0$ when $x = a$,

$\therefore$ $\text{X} = \text{Q}\cdot 0 = 0$, when x is replaced by a,

i.e., a satisfies the equation $\text{X} = 0$, and is a root of it. Q.E.D.

COR. 1. *Every factor of* X *that is itself a function of* x *may be put equal to* 0, *and the roots of the equations so formed are roots of the equation* $X = 0$.

COR. 2. *No rational integral equation* $X = 0$ *has more roots than the function* X *has linear factors* [factors of the form $x - a$]; *and if the equation be of the* n*th degree, it has not more than* n *roots.*

COR. 3. *If there be two rational functions of the same variable, neither of which is higher than the* n*th degree, and if these two functions be equal for more than* n *finite values of the variable; then are the two functions identical.*

Let $Ax^n + Bx^{n-1} + \cdots + Sx + T = A'x^n + B'x^{n-1} + \cdots + S'x + T'$
be a true equation for more than n finite values of x;

then will $A = A', \quad B = B', \quad \cdots \quad S = S', \quad T = T'$,

and $\quad Ax^n + Bx^{n-1} + \cdots + Sx + T \equiv A'x^n + B'x^{n-1} + \cdots + S'x + T'$.

For, if not, the equation

$$(A - A')x^n + (B - B')x^{n-1} + \cdots + (S - S')x + (T - T') = 0$$

has not more than n roots; [cr. 2

which is contrary to hypothesis;

$\therefore A = A', \quad B = B', \cdots$. Q.E.D.

NOTE. The roots may not be all different.

For if the function X have the same factor used two or more times, then the equation $X = 0$ is said to have two or more *equal roots*.

In general, if $X = (x - a)^p \cdot (x - b)^q \cdots$,

wherein $p, q, \cdots$ are positive integers such that $p + q + \cdots = n$;

then a is a *p-fold root*, b a *q-fold root*, and so on.

E.g., $x^3 - 3ax^2 + 3a^2x - a^3 \equiv (x - a)^3$,

and the three roots of the equation

$$x^3 - 3ax^3 + 3a^2x - a^3 = 0 \text{ are } a, a, a.$$

So, the equation $(x + a)^2 (x - b)^2 = 0$

has $-a, -a, b, b$ for its four roots.

It appears later that a set of equal roots are the limits of a set of unequal roots, and that if the equation $X = 0$ be of the nth degree, it has n roots, equal or unequal, real or imaginary.

THEOR. 5. *If, of the rational integral equation* $X = 0$, *the absolute term* $\doteq 0$, *some root* x' *of the equation* $\doteq 0$.

Let equation $X = 0$ be written $T = -Sx - Rx^2 - \cdots - Ax^n$,

and let $S, R, \cdots A$ stand fast, while x, T vary, and $T \doteq 0$;

then will some root $x' \doteq 0$.

For $\because D_{x'}T, = -S - 2Rx' - \cdots,$ [VII. ths. 13, 17

$\doteq -S$, a finite number; [$x' \doteq 0$

$\therefore$ the ratio $\text{inc}\,T : \text{inc}\,x'$ is finite; and the two infinitesimals are of the same order.

But $\because T = 0$ when $x' = 0$,

$\therefore$ if $T \sim 0$ be small, so is $x' \sim 0$;

i.e., $x' \doteq 0$ when $T \doteq 0$. Q.E.D.

COR. 1. *If the absolute term be* 0, *then is* 0 *a root of the equation.*

For, if 0 be put for x, the equation $X = 0$ is satisfied. Q.E.D.

COR. 2. *If* A, *the coefficient of the highest power of the unknown element in* X, $\doteq 0$, *then a root of the equation* $\doteq \infty$.

For, if x be replaced by y^{-1} in the equation $X = 0$, that equation takes the form

$$\frac{A}{y^n} + \frac{B}{y^{n-1}} + \frac{C}{y^{n-2}} + \cdots + \frac{R}{y^2} + \frac{S}{y} + T = 0,$$

whence $Ty^n + Sy^{n-1} + Ry^{n-2} + \cdots + Cy^2 + By + A = 0$; [mult. by y^n

and if the absolute term $A \doteq 0$,

then some root $y' \doteq 0$, and some root $x', \equiv 1 : y' \doteq \infty$. Q.E.D.

NOTE. If the last two, three, $\cdots$ of the coefficients $\cdots, R, S, T$ be zero, or approach zero, so do as many of the roots; and if the first two, three, $\cdots$ of the coefficients $A, B, C, \cdots$ approach zero, as many of the roots approach infinity.

E.g., if A, B, C, R, S be infinitesimals of the first order, T be zero, and D, Q be finite,

then three of the roots are infinites, each of the order $\frac{1}{3}$, two are infinitesimals, each of the order $\frac{1}{2}$, and one is zero.

§ 5. SIMPLE EQUATIONS INVOLVING ONE UNKNOWN.

PROB. 1. TO SOLVE A SIMPLE EQUATION INVOLVING ONE UNKNOWN ELEMENT.

Multiply both members of the equation by the l. c. mlt. of the denominators, if any. [th. 2

Transpose to one member all terms that involve the unknown element, and to the other member all other terms. [th. 1

Reduce both members to their simplest form, exhibiting or cancelling any common factors.

Divide both members by the coefficient of the unknown element. [II. ax. 5

To test the work, replace the unknown element by the result so found, in the original equation.

E.g., if $\frac{1}{6}(x+12)=\frac{1}{7}(6+3x)-\frac{1}{6}x$, [$x$ unk.

then $\because$ $7x+84=36+18x-7x$, [mult. by 42

and $(7-18+7)x=36-84$, [trans. $84, 18x, -7x$

i.e., $-4x=-48$, and $x=12$; [div. by -4

and $\because$ $\frac{1}{6}(12+12)\equiv\frac{1}{7}(6+36)-2$, [repl. x by 12

$\therefore$ 12 is the root sought.

So, if $\dfrac{(2a+b)b^2}{a(a+b)^2}x+\dfrac{a^2b^2}{(a+b)^3}=3cx+\dfrac{b}{a}x-\dfrac{3abc}{a+b}$,

then $\because$ $(2a+b)b^2(a+b)x+a^3b^2$
$=3ac(a+b)^3x+b(a+b)^3x-3a^2bc(a+b)^2$,

$\therefore$ $a^3b^2+3a^2bc(a+b)^2$
$=(3ac+b)(a+b)^3x-(2a+b)b^2(a+b)x$,

i.e., $a^2b[ab+3c(a+b)^2]=a(a+b)[ab+3c(a+b)^2]x$,

$$\therefore\ x=\frac{a^2b[ab+3c(a+b)^2]}{a(a+b)[ab+3c(a+b)^2]}=\frac{ab}{a+b}.$$

This value satisfies the given equation, and is the root sought.

NOTE 1. A simple equation can have but one root.

For any such equation may take the form $ax-b=0$, one linear factor. [th. 4 cr. 2

NOTE 2. Equations not simple sometimes reduce to simple equations, and may be solved like them.

E.g., if $\sqrt{x}-\sqrt{[x-\sqrt{(1-x)}]}=1$;

then $\because \sqrt{[x-\sqrt{(1-x)}]}=\sqrt{x}-1$, [trans. $\sqrt{x}$, change signs

$\therefore x-\sqrt{(1-x)}=x-2\sqrt{x}+1$, [sqr. both mem.

$\therefore \sqrt{(1-x)}=2\sqrt{x}-1$, [cancel x, change signs

$\therefore 1-x=4x-4\sqrt{x}+1$, [sqr. both mem.

$\therefore 4\sqrt{x}=5x$, [trans.

$\therefore 16x=25x^2$, [sqr. both mem.

$\therefore x(16-25x)=0$, [trans., factor

$\therefore x=0$ or $=\frac{16}{25}$.

Both of these results satisfy the given equation, and are roots.

For $\sqrt{\frac{16}{25}}-\sqrt{[\frac{16}{25}-\sqrt{(1-\frac{16}{25})}]}\equiv 1$ if the second radical be negative, and the other two positive;

and $\sqrt{0}-\sqrt{[0-\sqrt{(1-0)}]}\equiv 1$ if the last two radicals take their negative values.

But if the signs of the radicals be restricted, the equation may have no solution.

E.g., $\overset{+}{\sqrt{}}x-\overset{+}{\sqrt{}}[x-\overset{+}{\sqrt{}}(1-x)]=1$,

$\overset{-}{\sqrt{}}x-\overset{+}{\sqrt{}}[x-\overset{-}{\sqrt{}}(1-x)]=1$.

NOTE 3. GENERAL DISCUSSION: Every simple equation involving one unknown may be reduced to the form

$ax+b=a'x+b'$, whose general solution gives

$x=(b'-b):(a-a')$; and there are three cases:

(a) $a\neq a'$; then x has a single value, positive, negative, or zero, that satisfies the equation.

(b) $a=a'$, $b\neq b'$; then $x=\infty$.

This result may be interpreted in the language of limits by saying that if a, a' be variables, or either of them, and if $a\neq a'$ but $a\doteq a'$, then x grows larger and larger without bounds.

E.g., if A, A' travel along the same road in the same direction at a, a' miles an hour, and if A' be $(b'-b)$ miles ahead of A,

then the quotient $(b' - b) : (a - a')$ is the time before they will be together.

If the hourly gain, $a - a'$, be small, that time is long; if there be no gain, *i.e.*, if $a = a'$, they will never be together, and there is no value of x that satisfies the equation.

(*c*) $a = a'$, $b = b'$; then $x = 0 : 0$, and the equation is satisfied by any number whatever.

In the example above (*b*), A, A′ are now together and they will always be together.

§ 6. ELIMINATION.

THEOR. 6. *If there be two or more unknown elements and a system of two or more independent simultaneous equations that involve them, then the roots are not changed thereby if any one of these equations be replaced by the sum of this equation and any other or others of them.*

Let the equations $P = Q$, $P' = Q'$, $P'' = Q''$, ⋯ be any system of simultaneous equations, and for the equation $P = Q$ put the equation $P + P' = Q + Q'$,
or $P + P' + P'' = Q + Q' + Q''$, or ⋯; [II. ax. 2

then will the roots of the system of equations,
$P + P' + \cdots = Q + Q' + \cdots$, $P' = Q'$, $P'' = Q''$, ⋯,
be identical with the roots of the system first given.

For ∵ when $P = Q$, $P' = Q'$, $P'' = Q''$, ⋯, then also
$P + P' + P'' + \cdots = Q + Q' + Q'' + \cdots$;

∴ whatever set of values satisfy the equations
$P = Q$, $P' = Q'$, $P'' = Q''$, ⋯, the same values satisfy the equation $P' + P' + P'' + \cdots = Q + Q' + Q'' + \cdots$;
and conversely. Q.E.D.

COR. *In such a system of equations the roots are not changed if before the addition one or more of the equations be multiplied by any factor not a function of the unknown elements, and not* 0.

THEOR. 7. *If one equation of a system be solved for any one unknown element, in terms of the other unknown elements that enter into it, then the roots of the system are not changed thereby, if in the other equations this element be replaced by the value so found.*

For, let $P = Q$, $P' = Q'$, $P'' = Q''$, $\cdots$ be a system of equations involving $x, y, z, \cdots$ in any way; solve the equation $P = Q$ for x, giving $x = f(y, z, \cdots)$, and substitute this expression for x in the other equations, giving them the new forms $R' = S'$, $R'' = S''$, $\cdots$;

then $\because$ x and $f(y, z, \cdots)$ have the same values,

$\therefore$ whatever values of $x, y, z, \cdots$ make identities of the equations $P = Q$, $P' = Q'$, $P'' = Q''$, $\cdots$, the same values make identities of the equations $R' = S'$, $R'' = S''$, $\cdots$,

and, conversely, whatever values of $y, z, \cdots$ make identities of the equations $R' = S'$, $R'' = S''$, $\cdots$, the same values of $y, z, \cdots$ make identities of the equations

$P = Q$, $P' = Q'$, $P'' = Q''$, $\cdots$,

i.e., both systems have the same roots. Q.E.D.

NOTE. By aid of Theors. 1, 2, 3, 6 a system of n independent simultaneous equations containing n unknown elements may be reduced to a new system of $n-1$ equations, containing $n-1$ unknown elements, whose roots are identical with the roots of the original system, and these $n-1$ equations to $n-2$ equations, $\cdots$, to two equations, to one equation.

The process by which, one after another, the several unknowns are removed from the system of equations is a case of *elimination*. In general when from two or more (say n) given equations a new equation is got that is free from at least $n-1$ of their elements, those elements are *eliminated between the given equations;* and the new equation, or its first member when the second member is zero, is the *resultant* of the given equations.

The elimination is *reversible* when, whichever $n-1$ of the given equations, together with the resultant, were known to be true, the remaining equation would necessarily be likewise true; otherwise, the elimination is *irreversible*.

PROB. 2. TO ELIMINATE AN UNKNOWN ELEMENT FROM A PAIR OF EQUATIONS INVOLVING THE SAME TWO UNKNOWN ELEMENTS.

(*a*) *Simple equations.*

FIRST METHOD, ADDITION AND SUBTRACTION.

Find the least common multiple of the coefficients of that element which is to be eliminated; divide it, in turn, by these coefficients, and multiply the two equations through by their quotients.

Subtract one equation from the other, member from member.

E.g., to eliminate x from the pair of equations

$$6x+7y=85, \quad 2x+3y=33:$$

then $\because$ the l.c. mlt. of the coefficients 2, 6 is 6,

$\therefore\ 6x+7y=85, \quad 6x+9y=99,$ [mult. by 1, 3

$\therefore\ \quad 2y=14.$ [subtract

So, to eliminate x from the pair of equations

$$\text{L}x+\text{M}=0, \quad \text{L}'x+\text{M}'=0,$$

wherein L, M, L′, M′, are any expressions that do not contain x, but which may contain other unknown elements:

then $\because\ \text{LL}'x+\text{L}'\text{M}=0, \quad \text{LL}'x+\text{LM}'=0,$ [mult. by L′, L

$\therefore\ \text{LM}'-\text{L}'\text{M}\ =0.$ [subtract

NOTE. The work is often best arranged as follows:

Write the given equations under each other, and at the right, their respective multipliers with such signs that the new equation may be the algebraic sum of the products of the given equations by their multipliers. If there be two columns of multipliers, one to eliminate each unknown, write first the column to be first used.

When small, the partial products can be obtained and added mentally, and only the sums written down. Detached coefficients can be used in part of the work.

E.g., the first of the above examples becomes:

$$\begin{array}{lr|r|r}
 & 6x+7y=85 & 3 & -1 \\
 & 2x+3y=33 & -7 & 3 \\
\therefore & 4x \quad =255 & =24 & \\
 & -231 & & \\
 & 2y=14 & & \\
\therefore & x=6,\ y=7. & &
\end{array}
\qquad \text{or} \qquad
\begin{array}{ccc|cc}
6x+ & 7y= & 85 & 3 & -1 \\
2 & 3 & 33 & -7 & 3 \\
4 & & 24 & & \\
 & 2 & 14 & & \\
\multicolumn{3}{c}{x=6,\ y=7} & &
\end{array}$$

SECOND METHOD, COMPARISON.

Solve both equations for that element which is to be eliminated.

Put the two values thus found equal to each other.

E.g., to eliminate x from the pair of equations

$$6x+7y=85,\quad 2x+3y=33:$$

then $\quad x=\frac{1}{6}(85-7y)=\frac{1}{2}(33-3y),\qquad$ [sol. both eq. for x

THIRD METHOD, SUBSTITUTION.

Solve either equation for that element which is to be eliminated.

In the other equation replace this element by the value so found.

E.g., to eliminate x from the pair of equations

$$6x+7y=85,\quad 2x+3y=33:$$

then $\because\ x=\frac{1}{2}(33-3y),\qquad$ [sol. 2d eq. for x

$\therefore\ 99-9y+7y=85.\qquad$ [repl. x in 1st eq.

(*b*) *Equations of degree higher than the first.*

Of the three methods of elimination shown above (*a*) sometimes one, and sometimes another is most available.

In the pair of equations $l=ar^{n-1},\ s=\dfrac{ar^n-a}{r-1}\qquad$ [to elim. n

the method of substitution is best: multiply the first equation by r, and replace ar^n by lr in the other;

then $\quad s=\dfrac{lr-a}{r-1}.$

In the same pair of equations $\qquad$ [to elim. a

the method of comparison is best: solve both equations for a, and put the values equal;

then $\quad \dfrac{l}{r^{n-1}}=\dfrac{s(r-1)}{r^n-1}.$

In the pair of equations $x^2+y=11,\ y^2+x=7\qquad$ [to elim. x the first method is less easy; but the other two are available.

2. $\because\ x^2=11-y,\quad x^2=(7-y^2)^2=49-14y^2+y^4,$

$\therefore\ 11-y=49-14y^2+y^4.\qquad$ [comparison

3. $\because\ x=(7-y^2),$

$\therefore\ x^2=49-14y^2+y^4;$

$\therefore\ x^2+y=11$ gives $49-14y^2+y^4+y=11.\qquad$ [substitution

FOURTH METHOD, DIVISION.

Reduce the equations to the form P $= 0$, Q $= 0$, *wherein* P, Q *are functions of* x, y.

Divide P *by* Q, Q *by the remainder, and so on, as in finding the h. c. msr. of two entire numbers, until some remainder is found that is free from the element to be eliminated, or that has a common measure with* P, Q, *and the successive remainders.*

If this remainder do not contain such a common measure, equate it to 0 *for the resultant sought.*

If a common measure of P, Q *be found, divide each of them, or any two successive remainders, by this measure, and with the quotients proceed as before to find a resultant.*

If the solution of the given equations be sought, then:

Solve the resultant for the unknown element involved in it; replace this element in the next previous remainder by the values thus found; equate to 0, *and solve for the other unknown element.*

Equate to 0 *the common measure, if any, of* P, Q; *if the new equation thus found involve but one unknown element, solve it therefor; but if it involve both unknown elements, give to either of them any value whatever, and solve for the other.*

E.g., to eliminate y from equations P $= 0$, Q $= 0$, wherein

P is $x \cdot y^4 - \overline{2x+1} \cdot y^3 - \overline{x^3-x^2-x-1} \cdot y^2 + \overline{x^3-x^2-x+1} \cdot \overline{y+2}$,

and Q is $x \cdot y^3 + \overline{x^2-2x-1} \cdot y^2 - \overline{x^2+x-1} \cdot y - 2 \cdot \overline{x^2-x-1}$:

Divide P by Q; the remainder, R, $= x \cdot y^2 - \overline{2x+1} \cdot y + 2$;

so, divide Q by R; the remainder, S, $= \overline{x^2-1} \cdot \overline{y-2}$;

then $\because$ R, S have the h. c. msr. $y-2$,

and the quotients are $xy-1$, x^2-1,

$\therefore$ the resultant is $x^2-1=0$, whose roots are $^{\pm}1$,

and equations P$=0$, Q$=0$ are satisfied when, and only when, either $x = {}^{+}1$, ${}^{+}1 \cdot y - 1 = 0$; or $x = {}^{-}1$, ${}^{-}1 \cdot y - 1 = 0$; or $x =$ any value, $y = 2$.

So, to apply the fourth method to the pair of equations,

$$3x^2-4xy+5y^2-6x+7y=120, \quad 2x^2-3xy+5y^2=103:$$

Write P, Q in the form $\mathrm{L}x^2+\mathrm{M}x+\mathrm{N}$, $\mathrm{L}'x^2+\mathrm{M}'x+\mathrm{N}'$, wherein L, M, N, L′, M′, N′, may contain y but not x.

§ 7. SIMPLE EQUATIONS, TWO OR MORE UNKNOWNS.

PROB. 3. TO SOLVE A PAIR OF SIMPLE EQUATIONS, WHEREOF ONE HAS TWO UNKNOWN ELEMENTS, AND THE OTHER BUT ONE.

Solve that equation which has but one unknown element. [pr. 1

Replace this element by its value in the second equation, and solve for the other unknown element. [th. 4 cr. 2

E.g., to find x, y from the pair of equations

$$6x+7y=85, \quad 4x=24:$$

then $\quad x=6, \quad 36+7y=85, \quad y=7.$

PROB. 4. TO SOLVE A PAIR OF SIMPLE EQUATIONS, WHEREOF BOTH HAVE THE SAME TWO UNKNOWN ELEMENTS.

Combine the two equations so as to eliminate one unknown element, and form an equation involving the other unknown element.

Solve this equation for its unknown element, replace this element by its value in either of the given equations, and solve the equation so found for the other unknown element.

For a check, replace the two unknown elements by their values in either of the original equations.

E.g., to find x, y from the pair of equations

$$6x+7y=85, \quad 2x+3y=33;$$

then $\because \frac{1}{7}(85-6x)=\frac{1}{3}(33-2x),$ [elim. y

$\therefore\ 255-18x=231-14x,$

$\therefore\ -4x=-24, \quad x=6;$

$\therefore\ 36+7y=85, \quad y=7.$

So, to find x, y from the pair of equations

$$ax+by=c, \quad a'x+b'y=c':$$

then $\because ab'x+bb'y=cb', \quad ba'x+bb'y=bc',$ [II. ax. 4

$\therefore\ (ab'-a'b)\,x=(cb'-c'b), \quad x=\dfrac{cb'-c'b}{ab'-a'b},$

$\therefore\ a\cdot\dfrac{cb'-c'b}{ab'-a'b}+by=c, \qquad y=\dfrac{ac'-a'c}{ab'-a'b}.$

NOTE 1. The values of the two unknown elements may be got independently of each other, by separate eliminations; or else, having found one of them, the other may be written by symmetry.

E.g., if $ax + by = c$, $a'x + b'y = c'$, [above

then $\because$ these equations are not altered by interchanging a with b, a' with b', and x with y,

$\therefore$ x is the same function of a, b, a', b' as is y of b, a, b', a';

$\therefore$ the value of either x or y is found from that of the other by interchanging a with b and a' with b'.

So, if $x + y = a$, $x - y = b$:

then $\left\{ \begin{matrix} x \\ y \end{matrix} \right.$ is the half $\left\{ \begin{matrix} \text{sum} \\ \text{difference} \end{matrix} \right.$ of a, b.

NOTE 2. INCOMPATIBLE EQUATIONS: If two given equations be incompatible, no solution is possible.

E.g., the equations $2x + 3y = 13$, $2x + 3y = 15$ are incompatible;

for their resultant, $0 = 2$, is absurd.

NOTE 3. DEPENDENT EQUATIONS: If one equation be dependent on the other, and derivable from it, there is no single solution, but an infinite number of solutions.

E.g., the equations $2x + 3y = 13$, $6x + 9y = 39$ are but one equation in two forms, and any value may be given to either of the unknown elements, and the corresponding value of the other computed.

NOTE 4. GENERAL FORMULÆ: The two equations $ax + by = c$, $a'x + b'y = c'$ are the type-forms of every pair of two-unknown first-degree equations; their solution gives:

$$x = (cb' - c'b) : (ab' - a'b), \quad y = (ac' - a'c) : (ab' - a'b).$$

The solution of this pair of equations embraces the solution of all such pairs of equations: the reader may translate the formulæ into a practical rule for such solutions without the intermediate steps.

NOTE 5. GENERAL DISCUSSION: There are three general cases.

(*a*) $ab' \neq a'b$;

then x, y have single values, positive, negative, or zero, that satisfy both the equations.

(*b*) $ab' = a'b$, $cb' \neq c'b$;

then $x = \infty$, $y = \infty$.

For $\because ab' = a'b$, $cb' \neq c'b$, [hyp.

$\therefore a : a' = b : b'$, $c : c' \neq b : b'$, [II. ax. 13

$\therefore a : a' \neq c : c'$, [II. ax. 8

and $ac' \neq a'c$. Q.E.D. [II. ax. 13

And $\because ab' - a'b = 0$, $cb' - c'b \neq 0$, $ac' - a'c \neq 0$,

$\therefore x, = (cb' - c'b) : (ab' - a'b), = \infty$;

and $y, = (ac' - a'c) : (ab' - a'b), = \infty$. Q.E.D.

This result may be interpreted, in the language of limits, by saying that if a, a', b, b', be variables, or either of them, and if $ab' \neq a'b$ but $ab' \doteq a'b$, then x, y grow larger without bounds.

E.g., if $ax + by = c$, $a'x + b'y = c'$ be the equations of two straight lines,

then the values of x, y that satisfy both equations are the co-ordinates of the meeting-point of the two lines.

If $ab' \doteq a'b$, then $a : a' \doteq b : b'$, the two lines approach parallelism, the point of intersection recedes to a great distance, and the values of x, y become very great.

If $ab' = a'b$, then $a : a' = b : b'$, the two lines are parallel, they have no meeting-point, and there are no values of x, y that satisfy both equations.

(*c*) $ab' = a'b$, $cb' = c'b$;

then $x = 0 : 0$, $y = 0 : 0$,

and the equations are equivalent, and satisfied by giving any value to one unknown element and computing the corresponding value for the other.

For $\because ab' = a'b$, $cb' = c'b$, [hyp.

$\therefore a : a' = b : b'$, $c : c' = b : b'$,

$\therefore a : a' = c : c'$,

$\therefore ac' = a'c$. Q.E.D.

In the above example (*b*) the two lines, under the special conditions given in (*c*), are coincident, and every point is a common point. In general any value may be assumed at random for one co-ordinate, and the other may then be computed.

If $ab' = a'b$, there also appear the following special cases:

(*d*) $a' = 0,\ b' = 0,\ c' \neq 0$;

then $\quad x = -bc' : 0,\ y = ac' : 0$, which values are infinite.

(*e*) $a' = 0,\ b' = 0,\ c' = 0$;

then $\quad x = 0 : 0,\ y = 0 : 0$, which values are indeterminate.

(*f*) $b = 0,\ b' = 0,\ ac' \neq a'c$;

then $\quad x = 0 : 0,\ y = (ac' - a'c) : 0$, of which values one is indeterminate and the other infinite.

(*g*) $b = 0,\ b' = 0,\ ac' = a'c$;

then $\quad$ the equations are equivalent,

and $\quad x = 0 : 0 = c : a = c' : a'$,

and $\quad y = 0 : 0$, and is indeterminate.

(*h*) $a = 0,\ b = 0,\ b' = 0,\ a' \neq 0,\ c \neq 0$;

then $\quad x = 0 : 0,\ y = -a'c : 0$, of which values one is indeterminate and the other infinite.

(*i*) $a = 0,\ b = 0,\ b' = 0,\ c = 0$;

then $\quad x = c' : a',\ y = 0 : 0$, of which values one is determinate and the other indeterminate.

(*j*) $a = 0,\ a' = 0,\ b = 0,\ b' = 0$;

then $\quad x = 0 : 0,\ y = 0 : 0$, which values are indeterminate; but not both are finite unless $c = 0,\ c' = 0$.

The reader may interpret these results, and illustrate them by the meeting, when possible, of two straight lines.

Two important special cases appear when c, c' both vanish.

(*k*) $ab' \neq a'b,\ c = 0,\ c' = 0$;

then $\quad x = 0,\ y = 0$.

(*l*) $ab' = a'b,\ c = 0,\ c' = 0$;

then $\quad x = 0 : 0,\ y = 0 : 0$, which values are indeterminate.

The reader may interpret these results, and illustrate them by the meeting of two straight lines: he will observe that in both cases the two lines pass through the origin; in the first they meet there; in the second they coincide throughout.

PROB. 5. TO SOLVE A SYSTEM OF n INDEPENDENT SIMPLE EQUATIONS THAT INVOLVE THE SAME n UNKNOWN ELEMENTS.

Combine the n *equations, two and two, in* n − 1 *ways, so that each equation is used at least once, and so as to eliminate the same unknown element at each operation; thereby form* n − 1 *equations involving the same* n − 1 *unknown elements.*

So, combine these n − 1 *equations, and thereby form* n − 2 *equations involving the same* n − 2 *unknown elements; and so on till there results one equation involving but one unknown element.*

Solve this equation, and replace the unknown element by its value in one of the two equations involving two unknown elements.

Solve this equation for the second unknown element, and replace these two elements by their values in one of the three equations involving three unknown elements; and so on.

E.g., to find x, y, z from the system of equations

$$x+2y+3z=14,\quad 3x+2y+z=10,\quad 6x+9y+13z=63:$$

then $2x-2z=-4,\quad 15x-17z=-36,$ [elim. y fr. eqs. 1, 2; 2, 3

$\therefore\ 4x=4,$ [elim. z

$\therefore\ x=1,\quad y=2,\quad z=3.$ [repl. x, y

So, to find x, y, z, from the system of equations

$$ax+by+cz=d,\ a'x+b'y+c'z=d',\ a''x+b''y+c''z=d'':$$

then $\because\ x=\dfrac{d-by-cz}{a}=\dfrac{d'-b'y-c'z}{a'}=\dfrac{d''-b''y-c''z}{a''},$

$\therefore\ a'd-a'by-a'cz=ad'-ab'y-ac'z$

and $a''d'-a''b'y-a''c'z=a'd''-a'b''y-a'c''z,$ [elim. x

$$\therefore\ y=\frac{ad'-a'd+(a'c-ac')z}{ab'-a'b}=\frac{a'd''-a''d'+(a''c'-a'c'')z}{a'b''-a''b'},$$

$$\therefore\ (a'b''-a''b')(ad'-a'd)+(a'b''-a''b')(a'c-ac')z$$
$$=(ab'-a'b)(a'd''-a''d')+(ab'-a'b)(a''c'-a'c'')z,$$

$$\therefore\ z=\frac{(ab'-a'b)(a'd''-a''d')-(a'b''-a''b')(ad'-a'd)}{(ab'-a'b)(a'c''-a''c')-(a'b''-a''b')(ac'-a'c)}$$
$$=\frac{ab'd''+a'b''d+a''bd'-a''b'd-a'bd''-ab''d'}{ab'c''+a'b''c+a''bc'-a''b'c-a'bc''-ab''c'}.$$

The reader may write the values of x, y by symmetry.

NOTE 1. The equations must be so combined that no m of the $n-1$ equations got by the elimination of one unknown element shall represent less than $m+1$ of the original n equations; and that no m of the $n-2$ equations got by the elimination of two unknown elements shall represent less than $m+1$ of those $n-1$ equations; and so on. Otherwise the $m-1$ equations, or the $m-2$ equations, $\cdots$, will not be independent, and no determinate solution will finally be got.

NOTE 2. ALL THE UNKNOWN ELEMENTS NOT INVOLVED IN EVERY EQUATION. An unknown element that does not appear in any equation may be considered as already eliminated from it, and the work is shortened by so much. Those unknown elements that appear in the fewest equations may be eliminated first.

E.g., to find x, y, z, t, u from the system of equations

$$9x-2z+u=41, \qquad (1)$$

$$7y-5z-t=12, \qquad (2)$$

$$4y-3x+2u=5, \qquad (3)$$

$$3y-4u+3t=7, \qquad (4)$$

$$7z-5u=11: \qquad (5)$$

Of these equations, x appears in two, y in three, z in three, u in four, t in two.

Equations 1, 3 may be combined to eliminate x, and equations 2, 4 to eliminate t, and there result two new equations involving y, z, u.

These two equations may be combined to eliminate y, and there results one equation involving z, u.

This last equation may be combined with equation 5 to eliminate either z or u at pleasure.

NOTE 3. PARTICULAR ARTIFICES: The equations may have a symmetry, as to the unknown elements or functions of them, that permits shorter processes than those of the general rule. Sometimes the sum of the unknown elements, or of the functions of them, may be got first.

E.g., to find x, y, z, t, v from the system of equations

$$y+z+t+v=a, \quad (1)$$
$$z+t+v+x=b, \quad (2)$$
$$t+v+x+y=c, \quad (3)$$
$$v+x+y+z=d, \quad (4)$$
$$x+y+z+t=e; \quad (5)$$

then $\because$ $4x+4y+4z+4t+4v=a+b+c+d+e$, [add

$\therefore$ $x+y+z+t+v=\frac{1}{4}(a+b+c+d+e)$,

and $x=\frac{1}{4}(-3a+b+c+d+e)$, [sub. eq. 1

$y=\frac{1}{4}(a-3b+c+d+e)$, and so on.

So, to find x, y, z from the system of equations

$$\frac{1}{x}+\frac{1}{y}=\frac{4}{15},\quad \frac{1}{y}+\frac{1}{z}=\frac{11}{60},\quad \frac{1}{z}+\frac{1}{x}=\frac{1}{4};$$

then $\because$ $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{7}{10}$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{20}$, [add, div. by 2

$\therefore$ $\frac{1}{x}=\frac{7}{20}-\frac{11}{60}=\frac{1}{6}$, $\frac{1}{y}=\frac{7}{20}-\frac{1}{4}=\frac{1}{10}$, $\frac{1}{z}=\frac{7}{20}-\frac{4}{15}=\frac{1}{12}$;

$\therefore$ $x=6$, $y=10$, $z=12$.

NOTE 4. THE NUMBER OF EQUATIONS GREATER THAN THAT OF UNKNOWN ELEMENTS. So many equations as there are unknown elements may be taken at random, and solved. If the roots so found satisfy the remaining equations, the system is possible; but, if not, the system is impossible.

E.g., to find x, y from the system of three two-unknown equations $3x+7y=17$, $5x-2y=1$, $8x+y=10$:

Take the first two equations and solve;

then $\because$ $x=1$, $y=2$ in these two equations,

and $\because$ these roots satisfy the third equation,

$\therefore$ this system of equations is possible, and the roots are 1, 2.

But not possible is the system of equations

$$3x+7y=17,\quad 5x-2y=1,\quad 8x+y=12.$$

In general, if there be $m+n$ compatible equations, and only m unknown elements, there are n *equations of condition;* and the constants must have such relations that these equations of condition are all satisfied.

E.g., given the system of three two-unknown equations

$$ax+by=c,\quad a'x+b'y=c',\quad a''x+b''y=c'';$$

then $\because\ x=\dfrac{cb'-c'b}{ab'-a'b},\quad y=\dfrac{ac'-a'c}{ab'-a'b},$ [from first two eq.

$\therefore$ that $a''x+b''y=c'$ be a true equation,

$$a''\cdot\frac{cb'-c'b}{ab'-a'b}+b''\cdot\frac{ac'-a'c}{ab'-a'b}=c'' \text{ must hold true};$$

i.e., $\quad ab'c''-ab''c'+a'b''c-a'bc''+a''bc'-a''b'c=0$

is the required equation of condition, and establishes the necessary relation between the given constants.

By this process all the unknown elements are eliminated from the given equations. So, in general, from n equations $n-1$ unknown elements may be eliminated.

NOTE 5. THE NUMBER OF UNKNOWN ELEMENTS GREATER THAN THAT OF EQUATIONS. If there be $m+n$ unknown elements and only n equations, all compatible, to n of these elements arbitrary values may be given, and the roots of the m equations will contain these arbitrary values, or some of them, and be themselves arbitrary, and the equations are indeterminate.

E.g., to find x, y from the single two-unknown simple equation

$$2x+3y=12:$$

Put $\quad y=\cdots\ {}^-5,\ {}^-4,\ {}^-3,\ {}^-2,\ {}^-1,\ 0,\ {}^+1,\ {}^+2,\ {}^+3,\ {}^+4,\ {}^+5,\cdots,$

then $\quad x=\cdots 13\tfrac{1}{2},\ 12,\ 10\tfrac{1}{2},\ 9,\ 7\tfrac{1}{2},\ 6,\ 4\tfrac{1}{2},\ 3,\ 1\tfrac{1}{2},\ 0,{}^-1\tfrac{1}{2},\cdots;$

or put $\quad x=\cdots\ {}^-5,\ {}^-4,\ {}^-3,\ {}^-2,\ {}^-1,\ 0,\ {}^+1,\ {}^+2,\ {}^+3,\ {}^+4,\ {}^+5,\cdots;$

then $\quad y=\cdots{}^+7\tfrac{1}{3},{}^+6\tfrac{2}{3},\ {}^+6,{}^+5\tfrac{1}{3},{}^+4\tfrac{2}{3},{}^+4,{}^+3\tfrac{1}{3},{}^+2\tfrac{2}{3},\ {}^+2,{}^+1\tfrac{1}{3},\ {}^+\tfrac{2}{3},\cdots;$

So, a series of values are given to x increasing by 1, and there result a series of values for y decreasing by $\tfrac{2}{3}$. This may be illustrated geometrically, by taking x, y as the running coordinates of a point on a straight line whose equation is

$$2x+3y=12.$$

Such series are called arithmetic progressions. [XII. § 1

If the results be limited by the condition that they shall all be integers, or all positive integers, it may happen that there are very few such roots, and certain modifications may be made in the method of solution.

E.g., to find all possible pairs of positive integral roots that satisfy the single equation $2x+3y=12$:

then $x=6-y-\frac{1}{2}y$.

Put $\frac{1}{2}y=z$;

then $y=2z,\ x=6-3z$.

Put $z=\cdots,\ ^-3,\ ^-2,\ ^-1,\ 0,\ ^+1,\ ^+2,\ ^+3,\cdots$;

then $y=\cdots,\ ^-6,\ ^-4,\ ^-2,\ 0,\ ^+2,\ ^+4,\ ^+6,\cdots$,

and $x=\cdots,\ 15,\ 12,\ 9,\ 6,\ 3,\ 0,\ ^-3,\cdots$,

wherein $6,0$; $3,2$; $0,4$ are the only pairs of roots admissible.

The progressions are here noticeable again; that for y increases by 2, and that for x decreases by 3, and they both go on either way forever.

So, to find sets of positive integral values for x, y, z that satisfy the pair of equations

$x+2y+3z=22,\quad 3x-5y+2z=-2$:

then $\because\ 11y+7z=68$, [elim. x

$\therefore\ z=9-2y+\frac{1}{7}(5+3y)$. [solve for z

Put $t=\frac{1}{7}(5+3y)$;

then $y=2t-2+\frac{1}{3}(t+1)$.

Put $u=\frac{1}{3}(t+1)$;

then $t=3u-1$.

$\therefore\ y=2(3u-1)-2+u \qquad =7u-4$,

$z=9-2(7u-4)+3u-1 \qquad =16-11u$,

and $x=22-2(7u-4)-3(16-11u)=19u-18$.

Put $u=\cdots,\ ^-3,\ ^-2,\ ^-1,\ 0,\ 1,\ 2,\ 3,\cdots$; [dif. $^+1$

then $x=\cdots,\ ^-75,\ ^-56,\ ^-37,\ ^-18,\ 1,\ 20,\ 39,\cdots$, [dif. $^+19$

$y=\cdots,\ ^-25,\ ^-18,\ ^-11,\ ^-4,\ 3,\ 10,\ 15,\cdots$, [dif. $^+7$

$z=\cdots,\ 49,\ 38,\ 27,\ 16,\ 5,\ ^-6,\ ^-17,\cdots$, [dif. $^-11$

and the only set of positive integral roots is 1, 3, 5.

But of sets, whereof two are positive and one negative, or vice versa, there are an infinite number;

and of sets whereof all three are negative there are none.

§ 8. GRAPHIC REPRESENTATION OF SIMPLE EQUATIONS INVOLVING TWO UNKNOWNS.

EVERY simple equation involving two unknown elements may be reduced to the type-form, $y = mx + b$. [VII. § 11

E.g., the equation $\text{A}x + \text{B}y = \text{C}$ becomes $y = -\frac{\text{A}}{\text{B}}x + \frac{\text{C}}{\text{B}}$,

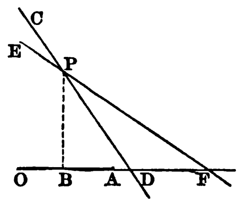
wherein $-\frac{\text{A}}{\text{B}} \equiv m$, $\frac{\text{C}}{\text{B}} \equiv b$.

Every such equation may therefore be represented by a straight line; and conversely, every straight line has its equation.

E.g., the equation $6x + 7y = 85$ reduces to $y = -\frac{6}{7}x + 12\frac{1}{7}$, wherein $-\frac{6}{7} \equiv m$, $12\frac{1}{7} \equiv b$, of the type-form.

This equation is represented by the line CD below.

This figure serves also to illustrate the solution of indeterminate equations [§ 7, nt. 5], wherein x is a variable and y a function of x.



If there be two simple equations involving the same two variables, and if it be required to find roots that satisfy both of them, then the two loci, platted with reference to the same origin, reference-line, and scale, will meet in a point whose co-ordinates are the roots sought.

E.g., if $6x + 7y = 12$, $2x + 3y = 4$ be a pair of simultaneous equations whose loci are CD, EF,

and if these loci meet at P;

then the lengths of the co-ordinates OB, BP are the common roots of the two equations.

But if the coefficients of the variables in one equation be nearly equal to those in the other, then the loci are nearly parallel, and the point of intersection may recede to a great distance; if they be identical with those in the other, or equimultiples of them, then the two loci are $\left\{\begin{matrix}\text{coincident}\\ \text{parallel}\end{matrix}\right.$ if the absolute term of the

first $\left\{\begin{matrix}\text{be}\\ \text{be not}\end{matrix}\right.$ the like multiple of that of the other; and there are $\left\{\begin{matrix}\text{an infinite number of}\\ \text{no}\end{matrix}\right.$ common roots.

If one of a pair of equations involving the same two variables be $y=0$, the locus of this equation is the line ox, and the solution of the pair of equations $y=0$, $y=mx+b$ reduces to the solution of the single simple equation involving one unknown element, $mx+b=0$, wherein the locus of x, y is the point where the pair of lines cross, and whose co-ordinates are $-\frac{b}{m}$, 0.

§ 9. BEZOUT'S METHOD, UNKNOWN MULTIPLIERS.

Let
$$\begin{aligned}&a_1x+b_1y+c_1z+\cdots=h_1,\\&a_2x+b_2y+c_2z+\cdots=h_2,\\&\cdots\quad\cdots\quad\cdots\quad\cdots,\\&a_nx+b_ny+c_nz+\cdots=h_n,\end{aligned}$$
be a system of n simple equations involving any same n unknown elements.

Multiply the first equation by k_1, the second by k_2, $\cdots$, the nth by k_n, wherein $k_1=0$, and $k_2, \cdots k_n$ are unknown;

then $\because$
$$\begin{aligned}&a_1x+b_1y+c_1z+\cdots=h_1,\\&k_2a_2x+k_2b_2y+k_2c_2z+\cdots=k_2h_2,\\&\cdots\quad\cdots\quad\cdots\quad\cdots,\\&k_na_nx+k_nb_ny+k_nc_nz+\cdots=k_nh_n,\end{aligned}$$
$$\begin{aligned}\therefore\ &(a_1+k_2a_2+\cdots+k_na_n)x\\&+(b_1+k_2b_2+\cdots+k_nb_n)y\\&+(c_1+k_2c_2+\cdots+k_nc_n)z+\cdots\\&=h_1+k_2h_2+\cdots+k_nh_n.\end{aligned}$$

Put all the coefficients except that of x equal to 0, *i.e.*, put $b_1+k_2b_2+\cdots+k_nb_n=0$, $c_1+k_2c_2+\cdots+k_nc_n=0, \cdots$, and thus form a system of $n-1$ equations involving the same $n-1$ unknown elements, $k_2, \cdots k_n$.

Whichever of $k_1, k_2, \cdots k_n$ be taken as 1, and the others as unknown, the ratios $k_1:k_2:\cdots k_n$ come out the same; but if k_2 or k_3 or $\cdots$ be ∞ when $k_1=1$, then k_2 or k_3 or $\cdots$ should be taken as 1, and $k_1, \cdots$ as unknown, whence $k_1=0$.

So, by aid of the multipliers $l_1 \cdots l_{n-1}$, reduce this system to a system of $n-2$ equations involving the same $n-2$ unknown elements, say $l_2 \cdots l_{n-1}$, and so on; and finally to two equations involving two unknown elements, say r_2, r_3, and to one equation involving one unknown element, say t.

Solve this equation for t, then solve for r_2, r_3, then for $\cdots$, then for $l_2 \cdots l_{n-1}$, then for $k_2 \cdots k_n$, then for $x, y, z, \cdots$.

E.g., to find x, y from the pair of equations

$$ax + by = c, \quad a'x + b'y = c':$$

then $\quad (a + ka')x + (b + kb')y = c + kc'$.

Put $\quad b + kb' = 0$;

then $\quad k = -\dfrac{b}{b'}$ and $x = \dfrac{c + kc'}{a + ka'} = \dfrac{cb' - c'b}{ab' - a'b}$.

So, put $(a + ka') = 0$;

then $\quad k = -\dfrac{a}{a'}$ and $y = \dfrac{c - kc'}{b - kb'} = \dfrac{ac' - a'c}{ab' - a'b}$.

So, to find x, y, z from the system of equations

$$ax + by + cz = d, \quad a'x + b'y + c'z = d', \quad a''x + b''y + c''z = d'':$$

then $\quad (a + k'a' + k''a'')x + (b + k'b' + k''b'')y + (c + k'c' + k''c'')z$
$= d + k'd' + k''d''$.

Put $\quad b + k'b' + k''b'' = 0, \quad c + k'c' + k''c'' = 0$; $\quad$ [k', k'' unk.

then $\quad b + hc + (b' + hc')k' + (b'' + hc'')k'' = 0$.

Put $\quad b'' + hc'' = 0$;

then $\quad h = -\dfrac{b''}{c''}$, and $k' = -\dfrac{b + hc}{b' + hc'} = \dfrac{b''c - bc''}{b'c'' - b''c'}$.

So, $\quad k'' = \dfrac{bc' - b'c}{b'c'' - b''c'}$.

But $\because x = \dfrac{d + k'd' + k''d''}{a + k'a' + k''a''}$,

$$\therefore x = \frac{d + \dfrac{b''c - bc''}{b'c'' - b''c'} \cdot d' + \dfrac{bc' - b'c}{b'c'' - b''c'} \cdot d''}{a + \dfrac{b''c - bc''}{b'c'' - b''c'} \cdot a' + \dfrac{bc' - b'c}{b'c'' - b''c'} \cdot a''}.$$

The reader may reduce this fraction to a simple fraction, and write the values of y, z by symmetry.

§ 10. SPECIAL PROBLEMS OF THE FIRST DEGREE.

In a special problem certain elements are given and certain other elements have given relations to those first named, and are to be found. These relations are the same whether expressed in ordinary language or in symbolic language. If in symbolic language, their expression gives an equation or a system of equations; and the elements whose values are to be found are the unknown elements of these equations.

The solution of a problem embraces three distinct parts: (1) putting it into equation; (2) solving the equation or system of equations; (3) discussing the results under special conditions.

A problem is of the first degree if its solution depend on the solution of an equation or system of equations of the first degree only.

Prob. 6. To put a special problem into equation.

By careful study of the enunciation of the problem, ascertain which of the elements named in it are known, and which are unknown; represent both the known and the unknown elements by symbols; and express in symbolic language all the relations that subsist between them.

These symbolic expressions are the equations sought.

Note 1. It may be convenient to express all the unknown elements by aid of a single symbol.

E.g., to divide $6341 among A, B, C, so that B shall have $420 more than A, and C $560 more than B:

Put x for A's share, $x+420$ for B's, $x+420+560$ for C's;

then $x+\overline{x+420}+\overline{x+420+560}=6321$, a single one-unknown simple equation.

So, to divide the number 144 into four parts, such that the first part increased by 5, the second decreased by 5, the third multiplied by 5, and the fourth divided by 5, shall all equal the same number:

Put x for the number to which the several results are equal;

then $\overline{x-5}+\overline{x+5}+\overline{x:5}+\overline{x\cdot 5}=144$.

NOTE 2. It may be convenient to express different unknown elements by different symbols; and to form a system of simultaneous equations involving two or more unknown elements.

E.g., a vintner at one time sells 20 dozen of port wine, and 30 dozen of sherry, and for the whole receives \$600; and at another time he sells 30 dozen of port and 25 dozen of sherry, at the same price as before, and for the whole receives \$700.

Put x for the price of a dozen of port, and y for that of a dozen of sherry;

then $20x + 30y = 600$, $30x + 25y = 700$, a pair of two-unknown simple equations.

So, if a certain rectangular bowling-green were 5 yards longer and 4 yards broader, it would contain 113 yards more; but if it were 4 yards longer and 5 yards broader, it would contain 116 yards more.

Put x, y for the length and breadth;

then $(x+5)\cdot(y+4) = xy + 113$, $(x+4)\cdot(y+5) = xy + 116$.

So, if A, B, C, D engage to do a certain piece of work; if A, B together can do it in 12 days; A, D in 15 days; C, D in 18 days; and if B, C begin the work, after 3 days A joins them, after 4 days more D joins them, and all working together they finish it in 2 days, in what time can each man do it working alone?

Put x, y, z, u for the number of days needed by A, B, C, D;

then $\frac{1}{x}+\frac{1}{y}=\frac{1}{12}$, $\frac{1}{x}+\frac{1}{u}=\frac{1}{15}$, $\frac{1}{z}+\frac{1}{u}=\frac{1}{18}$, $\frac{9}{y}+\frac{9}{z}+\frac{6}{x}+\frac{2}{u}=1$, a system of four simple four-unknown equations.

NOTE 3. DISCUSSION OF THE SOLUTION: To *discuss* the solution of a problem whose answer is numerical, is to try whether all the conditions of the problem are satisfied by all or any of the numbers that are found to satisfy the equations into which the problem was translated; and, if not, to observe what other conditions the unknown elements must satisfy besides those taken account of in putting the problem into equation.

To discuss the solution of a problem whose answer is literal is to observe between what limiting numerical values of the known elements the problem is possible; and whether any singularities or remarkable circumstances occur within these limits.

E.g., in a certain two-digit number the first digit is half the other, and if 27 be added to the number, the order of the digits is reversed; what is the number?

Put x for first digit, y for second digit;

then $\because\ 2x=y,\quad 10x+y+27=10y+x,$

$\therefore\ x=3,\quad y=6,$ the number is 36; and $36+27=63$.

Were this the statement: in a certain two-digit number, the first digit is half the other, and if 24 be added to the number, the order of the digits is reversed;

then $\because\ 2x=y,\quad 10x+y+24=10y+x,$

$\therefore\ x=2\frac{2}{3},\quad y=5\frac{1}{3},$ and the number is impossible.

The statement of the problem puts a limitation upon the values of x, y not expressed by the equation: they must be integers.

Were this the statement: of two numbers the first is half the second, and if to ten times the first the second and 24 be added, the sum is the sum of ten times the second added to the first; then the same equations as before would express the relations, and the values $2\frac{2}{3}$, $5\frac{1}{3}$ would satisfy all the conditions.

For $\quad 2\cdot 2\frac{2}{3}\equiv 5\frac{1}{3},\quad 10\cdot 2\frac{2}{3}+5\frac{1}{3}+24\equiv 10\cdot 5\frac{1}{3}+2\frac{2}{3}.$

And were this the statement: in a certain two-digit number the first digit is half the other, and if a be added to the number, the order of the digits is reversed;

then $\quad 2x=y,\quad 10x+y+a=10y+x,\quad x=\frac{1}{9}a,\quad y=\frac{2}{9}a;$

the special condition is imposed that a shall be a multiple of 9 not greater than 36 nor less than -36;

i.e., a is 36, 27, 18, 9, 0, $^-9$, $^-18$, $^-27$, $^-36$,

and the number is 48, 36, 24, 12, 0, $^-12$, $^-24$, $^-36$, $^-48$.

§ 11. QUADRATIC EQUATIONS INVOLVING ONE UNKNOWN.

PROB. 7. TO SOLVE AN INCOMPLETE QUADRATIC EQUATION.

Reduce the equation to the type-form $x^2 = q$, *and take the square root of both members; then* $x = \pm\sqrt{q}$.

E.g., to find x from the equation

$$\tfrac{1}{3}(x^2-10)+\tfrac{1}{10}(6x^2-100)=3x^2-65:$$

then $\because$ $10x^2-100+18x^2-300=90x^2-1950,$ [mult. by 30

$\therefore$ $-62x^2=-1550,$

$\therefore$ $x^2=25$ and $x=\pm 5.$

NOTE. There are two square roots, opposites of each other; they are both real if the q of the type-form be positive, and both imaginary if the q be negative.

PROB. 8. TO SOLVE A COMPLETE QUADRATIC EQUATION.

Reduce the equation to the type-form $x^2+px=q$.

Add $\frac{1}{4}p^2$ *to both members of the equation; take the square root; and solve the equations thus found.*

The result is of the form $x=-\frac{1}{2}p\pm\frac{1}{2}\sqrt{(p^2+4q)}$.

E.g., to find x from the equation $3x^2+9x=120$:

then $\because$ $x^2+3x=40,$ [div. by 3

$\therefore$ $x^2+3x+2\frac{1}{4}=42\frac{1}{4},$ [add $(\frac{3}{2})^2$, $=2\frac{1}{4}$

$\therefore$ $x+1\frac{1}{2}=\pm 6\frac{1}{2},$ [extr. sqr. rts. of both mem.

$\therefore$ $x=-1\frac{1}{2}\pm 6\frac{1}{2}=5$ or -8; and 5, 8 are both roots.

So, to find x from the equation $ax^2+bx+c=0$:

then $$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2},$$

and $$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a};$$ and both values are roots.

NOTE 1. DOUBLE SIGNS: Since either $x+p$ or $-(x+p)$ is a square root of $x^2+px+\frac{1}{4}p^2$, the given quadratic is satisfied as well when

$$-(x+p)=\tfrac{1}{2}\sqrt[\pm]{(p^2+4q)} \text{ as when } x+p=\tfrac{1}{2}\sqrt[\pm]{(p^2+4q)};$$

but this gives only the two values for x written above.

NOTE 2. DISCUSSION OF THE EQUATION $x^2+px=q$, FOUR SPECIAL CASES. The roots are:

$$-\tfrac{1}{2}p+\tfrac{1}{2}\sqrt{(p^2+4q)},\quad -\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{(p^2+4q)}.$$

(*a*) p *positive*, q *negative.*

Two real roots, both negative, if p^2+4q be positive.
Two real roots, both negative, equal to $-\tfrac{1}{2}p$, if $p^2+4q=0$.
Two imaginary roots, conjugates, if p^2+4q be negative.

(*b*) p, q *both negative.*

Two real roots, both positive, if p^2+4q be positive.
Two real roots, both positive, equal to $-\tfrac{1}{2}p$, if $p^2+4q=0$.
Two imaginary roots, conjugates, if p^2+4q be negative.

(*c*) p, q *both positive.*

Two real roots, the smaller positive, the larger negative.

(*d*) p *negative*, q *positive.*

Two real roots, the smaller negative, the larger positive.

NOTE 3. SUMS AND PRODUCTS OF ROOTS. The sum of the two roots is $-p$, and their product is $-q$.

The reader may prove.

NOTE 4. THE ABSOLUTE TERM, 0. If $q=0$, then of the equation $x^2+px=0$ the two roots are 0 and $-p$, both real.

NOTE 5. SOLUTION BY FACTORING. Write the equation

$x^2+px-q=0$ in the form

$x^2+px+\tfrac{1}{4}p^2-\tfrac{1}{4}(p^2+4q)=0$,

i.e., in the form $(x+\tfrac{1}{2}p)^2-\tfrac{1}{4}(p^2+4q)=0$;

then $\because$ $[x+\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{(p^2+4q)}]\cdot[x+\tfrac{1}{2}p+\tfrac{1}{2}\sqrt{(p^2+4q)}]=0$,

and $\because$ this product vanishes when, and only when, one of its factors vanishes,

$\therefore$ the roots of the equations

$x+\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{(p^2+4q)}=0$

and $x+\tfrac{1}{2}p+\tfrac{1}{2}\sqrt{(p^2+4q)}=0$ [th. 4 cr. 1

are the roots of the given equation.

$\therefore$ $x=-\tfrac{1}{2}p+\tfrac{1}{2}\sqrt{(p^2+4q)}$,

$x=-\tfrac{1}{2}p-\tfrac{1}{2}\sqrt{(p^2+4q)}$.

In practice the factoring is often made at sight.

E.g., to find x from the equation $x^2 - 5x + 6 = 0$:

then $\because x^2 - 5x + 6 = (x-2)(x-3)$, [factoring

$\therefore$ the roots are 2 and 3.

NOTE 6. GENERAL RULE. The rule for solving incomplete quadratic equations may be stated in a more general form:

Reduce the equation to the type-form $ax^2 + bx + c = 0$.

Multiply (or divide) both members of the equation by a, *or by any factor or multiple of* a *that shall make the coefficient of the first term a perfect square.*

Add to both members of the equation whatever is necessary to make the first member a perfect square, and take the square root.

Solve the simple equations thus found.

The rule in this form often avoids fractions. Both rules rest on that for finding the square root, and are the same in principle.

The solution of the equation gives $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$.

The reader may translate this formula into a working rule for finding the value of x without writing the intermediate steps.

E.g., to find x from the equation $3x^2 + 9x = 120$:

then $\because 9x^2 + 27x = 360$, [mult. by 3

and $9x^2 + 27x \,\underline{|\,3x + 4\tfrac{1}{2}}$ [sq. rt. of 1st mem.

$$\begin{array}{l|l} \multicolumn{2}{l}{9x^2} \\ \hline 6x + 4\tfrac{1}{2} & 27x \\ & 27x + 20\tfrac{1}{4} \end{array}$$

$\therefore 3x + 4\tfrac{1}{2} = \sqrt{380\tfrac{1}{4}} = \pm 19\tfrac{1}{2}$,

and $x = 5$ or -8;

or, by direct substitution in the formula,

$$x = \frac{-9 \pm \sqrt{(9^2 - 4 \cdot 3 \cdot -120)}}{2 \cdot 3} = 5 \text{ or } -8.$$

NOTE 7. DISCUSSION OF THE EQUATION $ax^2+bx+c=0$; THREE SPECIAL CASES: $c=0$, $b=0$, $a=0$.

(*a*) *If* c, *the absolute term, be* 0;

then the equation $ax^2+bx=0$ gives $x=0$ and $x=-b:a$, two real roots, whereof one is 0.

(*b*) *If* b, *the coefficient of the first power of* x, *be* 0;

then the equation $ax^2+c=0$ gives $x=\pm\sqrt{(-c:a)}$, two real roots, opposites, if a, c be of contrary signs; two imaginary roots, conjugates, if a, c be of the same sign.

(*c*) *If* a, *the coefficient of the second power of* x, *be* 0;

then $\because\ x=\dfrac{-b+\sqrt{(b^2-4ac)}}{2a},\quad \dfrac{-b-\sqrt{(b^2-4ac)}}{2a},$

$\therefore\ x=(-b+\sqrt{b^2}):0,\quad (-b-\sqrt{b^2}):0;$ $\qquad$ [$a=0$

i.e., $\quad x=-0:0,\ \ 2b:0$ if $\sqrt{b^2}$ be $+b$;

and $\quad x=-2b:0,\ \ 0:0$ if $\sqrt{b^2}$ be $-b$.

In either case there is an infinite and an indeterminate root.

But this indeterminate root may be determined.

For $\because\ a=0$,

$\therefore$ when $x\neq\infty$, the equation $ax^2+bx+c=0$ becomes $bx+c=0$, whose single root is $-c:b$.

It may also be determined by multiplying both terms of the fraction $\dfrac{-b\pm\sqrt{(b^2-4ac)}}{2a}$ by $-b\mp\sqrt{(b^2-4ac)}$;

then $$x=\frac{b^2-(b^2-4ac)}{2a[-b\mp\sqrt{(b^2-4ac)}]}=\frac{2c}{-b\mp\sqrt{(b^2-4ac)}}$$
$$=-c:b \text{ or } -c:0 \text{ when } a=0.$$

This case is especially important as showing the value of the limits of the roots of the equation when $a\doteq 0$; and it is to be noted that as $a\doteq 0$ one of the roots $\doteq\infty$, and the other $\doteq -c:b$.

This is also evident if the equation be written in the form

$x^{-1}(b+cx^{-1})=-a.$ $\qquad$ [div. eq. $bx+c=-ax^2$ by x^2

For, if $a\doteq 0$,

then $\quad$ either $x^{-1}\doteq 0$, and $x\doteq\infty$,

or $\quad b+cx^{-1}\doteq 0$, and $x\doteq -c:b$;

i.e., $\quad$ both ∞ and $-c:b$ satisfy the equation and are roots.

To determine whether the root $\doteq {}^+\infty$ or ${}^-\infty$, observe that

$\because$ the sum ax^2+bx, $=-c$, remains finite when ax^2 and bx each $\doteq\infty$, *i.e.*, when $x\doteq\infty$,

$\therefore$ ax^2, bx have opposite signs.

Divide ax^2, bx by ax;

then $\because$ x and $b:a$ have opposite signs,

$\therefore$ if a, b have $\left\{\begin{matrix}\text{the same}\\ \text{opposite}\end{matrix}\right.$ signs, the root $\doteq\left\{\begin{matrix}{}^-\infty\\ {}^+\infty\end{matrix}\right.$ when $a\doteq 0$.

The reader may further discuss the equation $ax^2+bx+c=0$, after the manner of Note 2, and show that the two roots are

$$\left\{\begin{matrix}\text{real and unequal}\\ \text{real and equal}\\ \text{imaginary}\end{matrix}\right. \quad \text{when} \quad \left\{\begin{matrix}b^2>4ac.\\ b^2=4ac.\\ b^2<4ac.\end{matrix}\right.$$

Of the real and unequal roots he may show which is the larger; and of all real roots he may show the conditions that make them positive or negative. He may also show that in every case the sum of the two roots is $-b:a$, and their product $c:a$, and that if x', x'' stand for the two roots,

then $\quad ax^2+bx+c\equiv a(x-x')(x-x'')$.

NOTE 8. EQUATIONS SOLVED AS QUADRATICS: Every equation of either of the following forms, or reducible thereto, is solved by aid of quadratics:

(*a*) $\quad ax^{2n}+bx^n+c=0$,

(*b*) $\quad (ax^{2n}+bx^n+c)^{2m}+p(ax^{2n}+bx^n+c)^m+q=0$,

(*c*) $\quad (ax^{2n}+bx^n+c)^{2m}\pm(ex^n+f)^{2m}=0$,

(*d*) $\quad (ax^{2n}+bx^n+c)^{2m}\pm(dx^{2n}+ex^n)^{2m}=0$,

wherein a, b, c, d, e, f, p, q are independent of x, and may be real or imaginary.

Whether a given equation $\text{P}=0$, whose degree is even, be of form (*a*), appears at once. If it be not, then to see whether it reduces either to form (*b*) or to form (*c*), find R, the entire part of the square root of P: if the remainder $\text{P}-\text{R}^2$ be of the form $p\text{R}+q$, the equation reduces to $\text{R}=\frac{1}{2}[-p+\sqrt{(p^2-4q)}]$, and if also R or some root of R be of the form $ax^{2n}+bx^n+c$, the equation reduces further to (*b*); or if $\text{P}-\text{R}^2$ be $\pm$ a perfect

square, s^2, the equation reduces to $R = S\sqrt{\mp 1}$, and perhaps to (c). Or, arrange P to ascending powers of x, and find R', so much of $\sqrt{P}$ that R'^2 has the degree of P as to x; then if $P - R'^2$ be $\pm$ a perfect square, S'^2, the given equation reduces to $R = S'\sqrt{\mp 1}$, and perhaps to (d).

E.g., if $9x^4 - 52x^2 + 64 = 0$;

then $\because\ 81x^4 - 468x^2 + 676 = 100,$ [mult. by 9, add 100

$\therefore\ 9x^2 = 26 \pm 10,$

$\therefore\ x^2 = 4 \quad \text{or} \quad \frac{16}{9},$

$\therefore\ x = \pm 2 \quad \text{or} \quad \pm\frac{4}{3}$: four real roots.

So, if $(9x^4 - 52x^2 + 80)^2 + 9(9x^4 - 52x^2 + 80) - 400 = 0$;

then $\because\ 4(9x^4 - 52x^2 + 80)^2 + 36(9x^4 - 52x^2 + 80) + 81 = 1681,$

$\therefore\ 2(9x^4 - 52x^2 + 80) = -9 \pm 41$

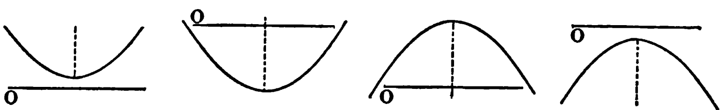
$= 32 \quad \text{or} \quad -50,$

$\therefore\ 9x^4 - 52x^2 + 80 = 16 \quad \text{or} \quad -25,$

$\therefore\ x = \pm 2,\ \pm\frac{4}{3},\ \pm\frac{1}{3}\sqrt{(26 \pm \sqrt{-209})}$: eight roots.

§ 12. GRAPHIC REPRESENTATION OF QUADRATIC FUNCTIONS.

Let $ax^2 + bx + c$ be any quadratic function of x, and put y equal to it; then different values may be given to x, the corresponding values of y computed, and the function platted. The plat is a parabola whose axis is vertical.



E.g., in the equation $y = x^2 + 2x - 3$.

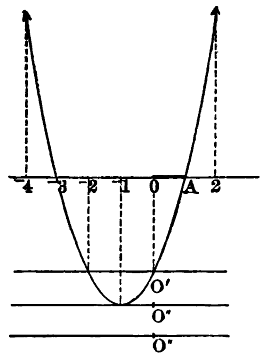
Put $x = \cdots, {}^-5, {}^-4, {}^-3, {}^-2, {}^-1, 0, {}^+1, {}^+2, {}^+3, {}^+4, {}^+5, \cdots$;

then $y = \cdots, 12, 5, 0, {}^-3, {}^-4, {}^-3, 0, 5, 12, 21, 32, \cdots$,

and the plat of the function is as shown in the figure, p. 319.

If there be a pair of equations involving x, y $y = x^2 + 2x - 3$, $y = 0$, their solution is reduced to the solution of a single quad-

ratic equation involving one unknown element, $x^2+2x-3=0$; and the roots of this equation are the abscissas of the points where the curve whose equation is $y=x^2+2x-3$ cuts the axis of abscissas whose equation is $y=0$.



The ordinates of the points of intersection are manifestly 0.

If the curve that represents the equation $y=x^2+2x-3$ remain fixed on the paper while the horizontal line that represents the equation $y=0$ moves downwards, taking in succession the positions $o'x'$, $o''x''$, $\cdots$, each ordinate of the curve is increased by the same length, and the value of y in the given equation is increased by the same number; and, by the simple change of the absolute term, the two roots of a quadratic equation may approach each other, then become equal, then imaginary.

E.g., of $x^2+2x=3$ the two roots are -3, 1,
of $x^2+2x=0$ the two roots are -2, 0,
of $x^2+2x=-1$ the two roots are -1, -1,
of $x^2+2x=-2$ the two roots are
$-1+\sqrt{-1}$, $-1-\sqrt{-1}$.

In all such cases it is said that a straight line cuts the curve in two points, real and separate, real and coincident, or imaginary, just as it is said that every quadratic equation has two roots, real and unequal, real and equal, or imaginary; and though it may seem strange to the beginner to say that one line cuts another in two points when it only touches it, or to say that it cuts it in two points when it does not cut it at all, yet the language and the demonstrations of Algebra gain greatly by this generality; and the pairs of roots so described have most of the algebraic properties of other pairs of roots: in particular, they each satisfy the given equation, and their sum is the $-p$ and their product the $-q$ of the type-form. [

PROB. 9. TO PLAT THE EQUATION $ax^2+bxy+cy^2=d$, USING NO IRRATIONAL FUNCTIONS OF x, y: a, b, c, d, x, y, ALL REAL.

(a) WHEN $b^2>4ac$, AND $cd \gtrless 0$.

Compute M, $\equiv \sqrt[+]{\dfrac{\pm cd}{b^2-4ac}}$; *and* N, $\equiv \sqrt[+]{\dfrac{\pm d}{c}}$.

To v, *an auxiliary variable, give any convenient series of values; and for each value of* v *find a pair of simultaneous values of* x, y *to satisfy the given equation:*

viz., $$x=\frac{v^2\mp 1}{2v}\cdot 2M,\quad y=\frac{v^2\pm 1}{2v}\cdot N-\frac{v^2\mp 1}{2v}\cdot\frac{bM}{c}.$$

Plat each of the points x, y; *and join them by a curve.*

(b) WHEN $4ac>b^2$: THEN ALWAYS $cd>0$. [x, y real

Compute M′, $\equiv \sqrt[+]{\dfrac{cd}{4ac-b^2}}$; *and* N′, $\equiv \sqrt[+]{\dfrac{d}{c}}$.

To the auxiliary variable v *give any convenient series of values; and for each of them find values of* x, y *to satisfy the equation:*

viz., $$x=\frac{2v}{1+v^2}\cdot 2M',\quad y=\frac{1-v^2}{1+v^2}\cdot N'-\frac{2v}{1+v^2}\cdot\frac{bM'}{c}.$$

Plat each of the points x, y; *and join them by a curve.*

(c) WHEN $b^2=4ac$: THEN ALWAYS $cd \not< 0$. [x, y real

Compute N′, $\equiv \sqrt[+]{\dfrac{d}{c}}$; *the plat is two parallel straight lines*

$$y=N'-\frac{b}{2c}x,\quad y=-N'-\frac{b}{2c}x.$$

(a) $\because$ $(2cy+bx)^2-(b^2-4ac)\cdot x^2=4cd$, [giv. eq.

$\therefore$ $[2cy+bx+x\sqrt{(b^2-4ac)}]\,[2cy+bx-x\sqrt{(b^2-4ac)}]$
$=4cd$;

$\therefore$ whatever value be given to v,

when $2cy+bx+x\sqrt{(b^2-4ac)}=\ 2\sqrt{(\pm cd)}\cdot v$,

then $2cy+bx-x\sqrt{(b^2-4ac)}=\pm 2\sqrt{(\pm cd)}:v$,

$\therefore$ $x=\dfrac{v^2\mp 1}{v}\cdot\sqrt{\dfrac{\pm cd}{b^2-4ac}}=\dfrac{v^2\mp 1}{v}\cdot M$; Q.E.D. [elim. y, solve

and $\because$ $4cy+2bx=2\dfrac{v^2\pm 1}{v}\cdot\sqrt{(\pm cd)}$, [add eqs. above

$\therefore\ y = \frac{v^2 \pm 1}{2v} \cdot \text{N} - \frac{v^2 \mp 1}{2v} \cdot \frac{b\text{M}}{c}.$ Q.E.D. [repl. x, sol. for y

(b) $\because\ (2cy + bx)^2 = 4cd - (4ac - b^2) \cdot x^2$ [giv. eq.

$= [2\sqrt{cd} + x\sqrt{(4ac - b^2)}][2\sqrt{cd} - x\sqrt{(4ac - b^2)}]$;

$\therefore$ whatever value be given to v,

when $2\sqrt{cd} + x\sqrt{(4ac - b^2)} = \frac{1+v}{1-v}(2cy + bx)$,

then $2\sqrt{cd} - x\sqrt{(4ac - b^2)} = \frac{1-v}{1+v}(2cy + bx)$,

$\therefore\ \frac{2\sqrt{cd} + x\sqrt{(4ac - b^2)}}{2\sqrt{cd} - x\sqrt{(4ac - b^2)}} = \left(\frac{1+v}{1-v}\right)^2$, [divide

$\therefore\ x = \frac{4v}{1+v^2}\sqrt{\frac{cd}{4ac - b^2}} = \frac{2v}{1+v^2} \cdot 2\text{M}'$; Q.E.D. [sol. for x

and $\because\ 4\sqrt{cd} = \frac{2 + 2v^2}{1 - v^2}(2cy + bx)$, [add eqs. above

$\therefore\ y = \frac{1 - v^2}{1 + v^2} \cdot \text{N}' - \frac{2v}{1 + v^2} \cdot \frac{b\text{M}'}{c}.$ Q.E.D. [repl. x, sol. for y

And $\because\ 4cd$ is the sum of the positive quantities

$(2cy + bx)^2$, $(4ac - b^2) \cdot x^2$,

$\therefore\ cd \not< 0.$ Q.E.D.

(c) $\because$ when $(2cy + bx)^2 = 4cd$

then $(2cy + bx - 2\sqrt{cd})(2cy + bx + 2\sqrt{cd}) = 0$,

$\therefore$ either $y = \sqrt{\frac{d}{c}} - \frac{bx}{2c}$ or $y = -\sqrt{\frac{d}{c}} - \frac{bx}{2c}$,

and conversely. Q.E.D.

E.g., to plat the equation $3x^2 + 5xy + 7y^2 = 425$. [fig., p. 322

Here $a, b, c, d = 3, 5, 7, 425$; $4ac - b^2 = {}^+59$,

and the case is (b);

$\text{M}' = \sqrt{(7 \cdot 425 : 59)} = 7.101$, $\text{N}' = \sqrt{(425 : 7)} = 7.792$,

$\frac{b\text{M}'}{c} = 5.072$;

and the coefficients of $2\text{M}', \text{N}', -\frac{b\text{M}'}{c}$ in the values of x, y are :

$$\begin{array}{llcccccccccccl}
\text{When} & v & =0 & \frac{\pm 1}{5} & \frac{\pm 1}{3} & \frac{\pm 1}{2} & \frac{\pm 2}{3} & \pm 1 & \frac{\pm 3}{2} & \pm 2 & \pm 3 & \pm 5 & \infty & \cdots, \\
\text{then} & \frac{2v}{1+v^2} & =0 & \frac{\pm 5}{13} & \frac{\pm 3}{5} & \frac{\pm 4}{5} & \frac{\pm 12}{13} & \pm 1 & \frac{\pm 12}{13} & \frac{\pm 4}{5} & \frac{\pm 3}{5} & \frac{\pm 5}{13} & 0 & \cdots, \\
\text{and} & \frac{1-v^2}{1+v^2} & =1 & \frac{12}{13} & \frac{4}{5} & \frac{3}{5} & \frac{5}{13} & 0 & \frac{-5}{13} & \frac{-3}{5} & \frac{-4}{5} & \frac{-12}{13} & -1 & \cdots,
\end{array}$$

$$\therefore\ x=\left(0,\frac{\pm 5}{13},\cdots,\frac{\pm 5}{13},\ 0\right)\times 14.202,$$

$$\text{and}\qquad y=\left(1,\frac{12}{13},\cdots,\frac{-12}{13},-1\right)\times 7.792-\left(0,\frac{\pm 5}{13},\cdots,\frac{\pm 5}{13},0\right)\times 5.072,$$

$$\begin{array}{llcccccl}
\textit{i.e.,} & x= & 0 & \pm 5.46 & \pm 8.52 & \pm 11.36 & \pm 13.11 & \pm 14.20\cdots, \\
\text{and} & y= & 7.79 & \left\{\begin{matrix}5.24\\9.14\end{matrix}\right. & \left\{\begin{matrix}3.19\\9.28\end{matrix}\right. & \left\{\begin{matrix}0.62\\8.73\end{matrix}\right. & \left\{\begin{matrix}-1.68\\7.68\end{matrix}\right. & \mp 5.07\cdots, \\
\text{for} & v= & 0 & \frac{\pm 1}{5} & \frac{\pm 1}{3} & \frac{\pm 1}{2} & \frac{\pm 2}{3} & \pm 1\ \cdots, \\
\text{and} & y= & -7.79 & \left\{\begin{matrix}-9.14\\-5.24\end{matrix}\right. & \left\{\begin{matrix}-9.28\\-3.19\end{matrix}\right. & \left\{\begin{matrix}-8.73\\-0.63\end{matrix}\right. & \left\{\begin{matrix}-7.68\\1.68\end{matrix}\right. & \cdots, \\
\text{for} & v= & \infty & \pm 5 & \pm 3 & \pm 2 & \frac{\pm 3}{2} & \cdots.
\end{array}$$

Give the same coefficients (except in order and sign) any values v_1, v_2 of v such that $v_2=-v_1$, or $=\pm 1 : v_1$, or $=\pm(1-v_1):(1+v_1)$, or $=\pm(1+v_1):(1-v_1)$. Such values are 0, ∞, ± 1; and $\pm\frac{1}{3}$, $\pm\frac{1}{2}$, ± 2, ± 3; and $\pm\frac{1}{5}$, $\pm\frac{2}{3}$, $\pm\frac{3}{2}$, ± 5.

So, to plat the equation

$$3x^2+5xy+7y^2-14x-51y=330:$$

here $\qquad 3(x+1)^2+5(x+1)(y-4)+7(y-4)^2=425,$

and the plat is as in the above example, except that the origin or datum-point to which the curve is referred will now be a unit to the right of, and 4 units below, the origin of the former plat.

NOTE. When $d=0$, no auxiliary v is needed: plat (a) reduces to a pair of lines $y=\frac{1}{2}[-b+\sqrt{(b^2-4ac)}]c^{-1}x$ and $y=\frac{1}{2}[-b-\sqrt{(b^2-4ac)}]c^{-1}x$; plat (b) to a point $x=0$, $y=0$; plat (c) to two coincident lines $y=-\frac{1}{2}bc^{-1}x$.

§ 13. SOLUTION OF QUADRATIC EQUATIONS BY AID OF CONTINUED FRACTIONS.

PROB. 10. TO SOLVE A QUADRATIC EQUATION BY AID OF CONTINUED FRACTIONS.

First root: Write the equation in the form $x(p+x)=q$;

then $$x=\frac{q}{p+x}=\frac{q}{p+\dfrac{q}{p+x}}=\frac{q}{p+\dfrac{q}{p+\dfrac{q}{p+x}}}=\cdots,$$

and the convergents are:

$$\frac{q}{p},\ \frac{pq}{p^2+q},\ \frac{p^2q+q^2}{p^3+2pq},\ \frac{p^3q+2pq^2}{p^4+3p^2q+q^2}=\cdots.$$

Second root: Write the equation in the form $x^2=-px+q$.

then $$x=-p+\frac{q}{x}=-p-\frac{q}{p-\dfrac{q}{x}}=-p-\frac{q}{p+\dfrac{q}{p-\dfrac{q}{x}}}=\cdots,$$

and the convergents are:

$$-p,\ -\frac{p^2+q}{p},\ -\frac{p^3+2pq}{p^2+q},\ -\frac{p^4+3p^2q+q^2}{p^3+2pq}=\cdots.$$

Of these two sets of convergents, when taken two and two in order, the products are $-q$, and the sums approach $-p$.

E.g., to find x from the equation $x^2+5x=2$;

then the two roots are $\dfrac{2}{5+\dfrac{2}{5+\dfrac{2}{5+\cdots}}}$ and $-5-\dfrac{2}{5+\dfrac{2}{5-\cdots}}$,

and the convergents are:

$$\frac{2}{5},\ \frac{10}{27},\ \frac{54}{145},\ \frac{290}{779},\ \cdots,\ \text{and}\ -5,\ -\frac{27}{5},\ -\frac{145}{27},\ -\frac{779}{145},$$

whose products taken two and two are all -2,

and whose sums so taken are $-4\frac{3}{5}$, $-5\frac{4}{135}$, $-4\frac{3907}{3915}$, $\cdots$,

that differ from -5 by $\dfrac{2}{5}$, $\dfrac{4}{135}$, $\dfrac{8}{3915}$, $\cdots$.

The reader may find the approximate values of x from the equation $ax^2+bx+c=0$, and translate the formulæ so found into words. In particular, he may find the approximate values of x when $a \doteq 0$; and show how the convergence of the continued fraction depends upon the reality of the roots. This form of solution by continued fractions applies only to quadratics; another form is given in XIII.

§ 14. MAXIMA AND MINIMA.

If x be a variable, and y be a function of x, and if as x increases, y increase for a time and then decrease, the greatest value that y thus attains is a maximum; but if as x increases, y decrease for a time and then increase, the least value that y thus attains is a minimum. So for any two variables.

The *normal* $\left\{\begin{matrix}\text{maxima}\\ \text{minima}\end{matrix}\right.$ of any function u of one or more independent variables $x, y, \ldots$ are such values of u that, if u were a little $\left\{\begin{matrix}\text{greater,}\\ \text{less,}\end{matrix}\right.$ some of $x, y, \ldots$ would become imaginary: they depend upon $x, y, \ldots$ being restricted to real values. *Abnormal* maxima and minima arise from other restrictions: as in the example below, where a certain rectangle is restricted to have its corners at or between the vertices of a certain triangle.

If, by solving a quadratic or otherwise, the relation of u to $x, y, \ldots$ be expressed in the form $P + Q\sqrt{U} = 0$, wherein P, Q are rational functions of $x, y, \ldots$ and U a function of u; then is u a normal $\left\{\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right.$ whenever its value is such that U vanishes and is not itself a maximum or minimum, but is $\left\{\begin{matrix}\text{a decreasing}\\ \text{an increasing}\end{matrix}\right.$ function of u; for a slight further $\left\{\begin{matrix}\text{increase}\\ \text{decrease}\end{matrix}\right.$ in the value of u makes $\sqrt{U}$ imaginary, while its equal $-P : Q$, a rational function of $x, y, \ldots$, remains real.

Theor. 8. *Maximum and minimum values of a continuous function occur alternately.*

For $\because$ just after passing through the first maximum value the function is decreasing,

and just before passing through the second maximum value the function is increasing; [df. max., min.

and $\because$ in passing from a decreasing state to an increasing state the function must pass through a minimum value; [df.

$\therefore$ between two maximum values lies at least one minimum value. Q.E.D.

So between two minimum values lies at least one maximum value. Q.E.D.

PROB. 11. TO DETERMINE MAXIMA AND MINIMA BY SOLVING QUADRATIC EQUATIONS.

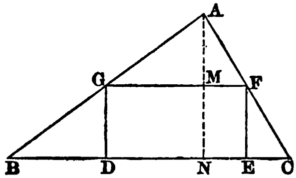
By an equation express the relation between x, *a variable, and* u, *a function of* x *to be maximized or minimized.*

Solve for x; *and if the value of* x *thus found involve an even root of a function of* u, *equate that function to* 0 *and solve for* u.

See whether the values of u *so found be maxima or minima.*

E.g., in a triangle to inscribe a rectangle of maximum area:

1. There is such a rectangle.



For let ABC be any triangle, AN its altitude, BC its base, DEFG any rectangle inscribed in it.

then ∵ DEFG approaches a minimum value, zero, as GF approaches A, and another minimum value, zero, as GF approaches BC,

for some intermediate position of GF there is a maximum value of DEFG. [th. 8

2. Let $u \equiv$ area DEFG, $x \equiv$ DE, $y \equiv$ DG, $b \equiv$ BC, $h \equiv$ AN;

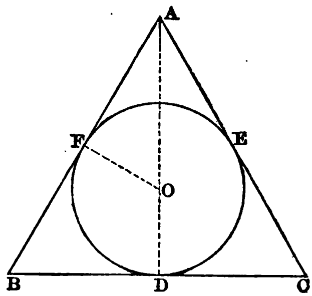
then ∵ $xy = u$, $x : h - y = b : h$,

∴ $u = by(h-y) : h$, $y = \frac{1}{2}h \pm \sqrt{\frac{1}{2}}[(bh^2 - 4hu) : b]$,

∴ the maximum value of u is $\frac{1}{4}bh$, and GF lies half-way between the vertex and base.

So, about a sphere to circumscribe a cone of minimum volume:

1. There is such a cone.



For let DEF be any circle and ABC an isosceles triangle circumscribed about it and tangent to it at D, E, F; let AD be the axis of the triangle, and let the whole figure revolve about AD;

then ∵ as the point A recedes from the circle, the lines AB, AC approach parallelism, and the triangle ABC grows larger and larger without bounds,

∴ the cone ABC grows larger and larger without bounds.

And $\because$ as the point A approaches the circle the lines AB, AC approach parallelism with BC, and the triangle ABC again grows larger and larger without bounds,

$\therefore$ the cone ABC grows larger and larger without bounds;

$\therefore$ for some intermediate position of A there is a minimum value of the volume of the cone. [th. 8

2. Let $V \equiv$ volume of cone ABC, $y \equiv$ AD, its altitude, $x \equiv$ BD, radius of base, $r \equiv$ radius of sphere;

then $V = \frac{1}{3}\pi x^2 y$;

and $\because$ $AB \cdot AF = AD \cdot AO$, $AB^2 = AD^2 + BD^2$, $AF^2 = AO^2 - OF^2$, $AO = AD - OD$, [geom.

$\therefore$ $(y^2 + x^2) \cdot (\overline{y - r}^2 - r^2) = y^2 \cdot (y - r)^2$,

$\therefore$ $x^2 y = r^2 y^2 : (y - 2r)$,

$\therefore$ $V = \frac{1}{3}\pi r^2 y^2 : (y - 2r)$,

$\therefore$ $y = [3V \pm \sqrt{(9V^2 - 24\pi r^3 V)}] : 2\pi r^2$;

and $\because$ that y be real, $9V^2 \not< 24\pi r^3 V$,

$\therefore$ the minimum value of V is $\frac{8}{3}\pi r^3$,

and the corresponding values of y, x are $4r$, $r\sqrt{2}$,

i.e., the minimum circumscribed cone has its altitude double the diameter of the sphere, the area of its base two great circles, its volume double the volume of the sphere, and its whole surface double that of the sphere: as the reader may prove.

So, to ascertain if the fraction $\dfrac{x^2 - 2x + 21}{6x - 14}$ have any limitations on its value, for real values of x:

Let $y \equiv \dfrac{x^2 - 2x + 21}{6x - 14}$;

then $\because$ $x = 1 + 3y \pm 3\sqrt{[(y - 2)(y + \frac{10}{9})]}$, [sol. for x

$\therefore$ that x be real, the factors $y - 2$ and $y + \frac{10}{9}$ must have the same sign,

i.e., y may not lie between 2 and $-\frac{10}{9}$, but may have any other value,

$\therefore$ 2 is a minimum and $-\frac{10}{9}$ a maximum value of x.

The reader may plat this function [a hyperbola], and the meaning of these statements will then be clearer to him.

So, to find the limitations on the value of the quadratic function ax^2+bx+c for real values of x:

Let $y \equiv ax^2+bx+c$;

then $\because$ $x=[-b \pm \sqrt{(b^2-4ac+4ay)}]:2a$,

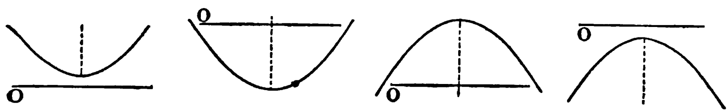
$\therefore$ for real values of x, $b^2-4ac+4ay$ cannot be negative,

i.e., $y+\dfrac{b^2-4ac}{4a}$ cannot be $\left\{\begin{matrix}\text{negative}\\\text{positive}\end{matrix}\right.$ when a is $\left\{\begin{matrix}\text{positive,}\\\text{negative,}\end{matrix}\right.$

$\therefore$ $y\left\{\begin{matrix}\nless\\\ngtr\end{matrix}\right.\dfrac{4ac-b^2}{4a}$, and $\dfrac{4ac-b^2}{4a}$ is its $\left\{\begin{matrix}\text{minimum}\\\text{maximum}\end{matrix}\right.$ value.

The plat of the function is a parabola whose axis is vertical and vertex $\left\{\begin{matrix}\text{downward}\\\text{upward}\end{matrix}\right.$ when a is $\left\{\begin{matrix}\text{positive}\\\text{negative}\end{matrix}\right.$; and this parabola $\left\{\begin{matrix}\text{does not}\\\text{does}\end{matrix}\right.$ cut the axis of x when b^2-4ac is $\left\{\begin{matrix}\text{negative.}\\\text{positive.}\end{matrix}\right.$

The four cases are represented by the four cuts below.



So, if $y \equiv \dfrac{ax^2+bx+c}{a'x^2+b'x+c'}$;

then $\because$ $x=\dfrac{-(b-b'y)}{2(a-a'y)}$

$$\pm\frac{\sqrt{[(b'^2-4a'c')y^2-2(bb'-2ac'-2a'c)y+b^2-4ac]}}{2(a-a'y)},$$

$\therefore$ that x be real,

$$(b'^2-4a'c')y^2-2(bb'-2ac'-2a'c)y+b^2-4ac \nless 0.$$

Write this quadratic function of y in the form

$$(b'^2-4a'c')(y-\alpha)(y-\beta),$$

wherein α, β are the roots of the equation got by putting this function equal to 0;

then three special cases are to be noted:

(*a*) $b'^2 - 4a'c'$ *positive.*

If α, β be real and $\alpha < \beta$, then, that x may be real, y must not lie between α, β; *i.e.*, y has α for a maximum and β for a minimum value.

If α, β be real and equal, or imaginary, the product $(y-\beta)(y-\alpha)$ is always positive, and there is no limitation on the value of y.

(*b*) $b'^2 - 4a'c'$ *negative.*

If α, β be real and $\alpha < \beta$, then, that x may be real, y must lie only between α, β; *i.e.*, y has α for a minimum and β for a maximum value.

If α, β be real and equal, or imaginary, then the product $(b'^2 - 4a'c')(y-\alpha)(y-\beta)$ is negative, and no real value of x is possible except for the particular values $y = \alpha = \beta$.

(*c*) $b'^2 - 4a'c'$ *zero.*

Then the quadratic function in y reduces to the form $py + q$; and that x be real this function may not be negative; and if p be $\left\{\begin{matrix}\text{positive},\\ \text{negative},\end{matrix}\right.$ $y + \frac{q}{p}$ cannot be $\left\{\begin{matrix}\text{negative};\\ \text{positive};\end{matrix}\right.$

$\therefore\ y \left\{\begin{matrix}\nless\\ \ngtr\end{matrix}\right. -\frac{q}{p}$, and its $\left\{\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right.$ value is $-\frac{q}{p}$.

NOTE. It is sometimes better not to solve for the independent variable, but to express in terms of it the function to be maximized or minimized: noting that if a be a positive constant, m, n odd positive integers, ϕ an increasing function, and ψ a decreasing function, then when u is a $\left\{\begin{matrix}\text{maximum},\\ \text{minimum},\end{matrix}\right.$ so are $u \pm a,\ au,\ u^{\frac{m}{n}},\ \phi(u)$; but $\pm a - u,\ a : u,\ u^{-\frac{m}{n}},\ \psi(u)$ are $\left\{\begin{matrix}\text{minima.}\\ \text{maxima.}\end{matrix}\right.$

E.g., to divide a real number $2a$ into two real parts whose product is a maximum:

Let $a - z$ and $a + z$ be the two parts;

then $\because\ (a-z)\cdot(a+z) = a^2 - z^2$,

and $\because$ the product $a^2 - z^2$ is greatest when $z = 0$,

$\therefore$ the parts are a and a. Q.E.D.

There is no minimum;

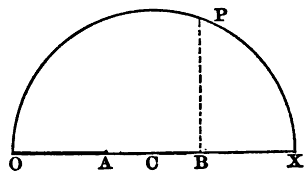
for, as z grows larger, $a^2 - z^2$ grows less without bounds.

Let $a - z \equiv x$;

then $a + z = 2a - x$, and the product $a^2 - z^2 = x(2a - x)$.

To plat the locus of the equation $y^2 = x(2a - x)$:

Take OX $= 2a$, and on OX as a diameter describe a semicircle; take B any point on OX, and draw BP perpendicular to OX;



then $\because$ $\text{BP}^2 = \text{OB} \cdot \text{BX}$, [geom.

and $y \equiv \text{BP}$, $x \equiv \text{OB}$, $2a - x \equiv \text{BX}$,

$\therefore$ the semicircle is the locus sought,

and y^2 is greatest when B is at the centre,

i.e., when OB, BX, BP, each $= a$.

So, to resolve a real number a^2 into two real positive factors whose sum is a minimum:

Let x, y be the two parts;

then $\because$ $(x + y)^2 = (x - y)^2 + 4xy = (x - y)^2 + 4a^2$,

$\therefore$ $(x + y)$ is least when $x - y = 0$, *i.e.*, when $x = y = a$.

There is no maximum;

for, as $x \sim y$ grows larger without bounds, so does $x + y$.

From these two examples it appears that of all rectangles with the same perimeter the square has the greatest area, and that of all rectangles with the same area the square has the least perimeter. So, often the same conditions that make a variable u a maximum or minimum when some other variable v is constant, also make v a maximum or a minimum when u is constant.

Other maxima and minima may be found by aid of the above.

E.g., to make $\dfrac{x}{ax^2 + b}$ a maximum: [a, b positive

Make the reciprocal, $ax + b : x$, a minimum;

then $\because$ the product of ax and $b : x$ is the constant ab,

$\therefore$ their sum is a minimum at $2\sqrt{ab}$;

i.e., the given function is a maximum at $\frac{1}{2}a^{-\frac{1}{2}}b^{-\frac{1}{2}}$.

§ 15. SIMULTANEOUS EQUATIONS.

Prob. 12. To solve two equations involving the same two unknown elements when one of the equations is simple.

Eliminate one of the unknown elements. [pr. 2

Solve the resultant for the other unknown element and replace this element by its value in the simple equation.

Solve this equation for the first unknown element.

E.g., to find the values of x, y from the pair of equations

$$3x + 2y = 20, \quad 3x^2 + 5xy + 7y^2 = 425:$$

then $\because\ x = \frac{1}{3}(20 - 2y)$, [sol. first eq. for x

$\therefore\ \frac{1}{3}(20-2y)^2 + \frac{5}{3}y(20-2y) + 7y^2 = 425$, [repl. x in sec. eq.

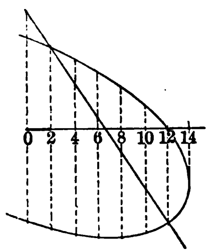
$\therefore\ 15y^2 + 20y = 875$, and $y = 7$ or $-8\frac{1}{3}$, [sol. quad. for y

$\therefore\ 3x + 2\cdot 7 = 20$, and $x = 2$, [repl. y in first eq.

or $3x - 2\cdot 8\frac{1}{3} = 20$, and $x = 12\frac{2}{9}$,

and the two pairs of roots are: $2, 7;\ 12\frac{2}{9}, -8\frac{1}{3}$.

That both pairs of roots satisfy the two equations appears by direct substitution, and that there ought to be two pairs of roots is evident from the plat.



Let $x = 0, 2, 4, 6, 8, 10$;

then in the equation $3x + 2y = 20$

$y = 10, 7, 4, 1, {}^-2, {}^-5$,

and in the eq'n $3x^2 + 5xy + 7y^2 = 425$

$y = {}^+7.8, {}^+7, {}^+6.2, {}^+4.9, {}^+3.5, {}^+1.5$,

or $y = {}^-7.8, {}^-8.2, {}^-8.9, {}^-9.2, {}^-9.3, {}^-8.6$,

Note 1. The two pairs of roots may coincide.

E.g., to find x, y from the pair of equations

$$7x^2 + 6xy + 8y^2 + 12x + 16y - 88 = 0, \quad 23x + 22y = 68:$$

then the resultant is $y^2 - 2y + 1 = 0$, and $x, y = 2, 1;\ 2, 1$.

The geometric interpretation of the equality of the roots is that the loci represented intersect in coincident points; *i.e.*, that they are tangent. A slight change in either equation so changes the locus that the points separate or disappear. Then the two roots are real and separate, or imaginary.

NOTE 2. Special expedients may be useful.

E.g., to find x, y from the pair of equations

$$x+y=7, \quad xy=12:$$

then $\because$ $(x-y)^2=(x+y)^2-4xy=1$,

$\therefore$ $x-y=\pm 1$;

and $\because$ $2x=(x+y)+(x-y)$, $2y=(x+y)-(x-y)$,

$\therefore$ $x=4$ or 3, $y=3$ or 4;

and the two pairs of roots are: 4, 3; 3, 4.

So, to find x, y from the pair of equations

$$x^2+y^2=125, \quad x-y=5:$$

then $\because$ $(x+y)^2+(x-y)^2=2(x^2+y^2)$,

$\therefore$ $(x+y)^2=225$ and $x+y=\pm 15$,

$\therefore$ $x=10$ or -5, $y=5$ or -10,

and the two pairs of roots are: 10, 5; -5, -10.

PROB. 13. TO SOLVE TWO EQUATIONS INVOLVING THE SAME TWO UNKNOWN ELEMENTS WHEN BOTH EQUATIONS ARE QUADRATIC.

Eliminate one of the unknown elements by division; solve the resultant biquadratic equation for the other.

Replace this element by each of the four roots so found, in the equation formed by equating to zero any remainder or divisor that contains the other unknown element in the first degree, and solve for that element.

E.g., to find the four values of x, y from the pair of equations

$$2xy+5y^2-195=0, \quad 3x^2-4xy-7=0: \quad \text{then}$$

$$\begin{array}{r|l|l}
\underline{2y\cdot x+(5y^2-195)} & 3x^2-4y\cdot x-7 & \underline{3x-(23y^2-582)} \\
 & \underline{2y\qquad\qquad\qquad\qquad} & \\
 & 6y\cdot x^2-8y^2\cdot x-14y & \\
 & \underline{6y\cdot x^2+3(5y^2-195)x} & \\
 & -(23y^2-585)x-14y & \\
 & \underline{2y\qquad\qquad\qquad\qquad\qquad} & \\
 & -2y(23y^2-585)x-28y^2 & \\
 & \underline{-2y(23y^2-585)x-(5y^2-195)(23y^2-585)} & \\
 & (5y^2-195)(23y^2-585)-28y^2 &
\end{array}$$

Equate this last remainder to zero;

then $\because$ $115y^4 - 7438y^2 + 114075 = 0$,

$$\therefore\ y^2 = 25 \text{ or } \frac{4563}{115},$$

$$\therefore\ y = \pm 5 \text{ or } \pm\frac{117}{\sqrt{345}},$$

and $$x = \pm 7 \text{ or } \mp\sqrt{\frac{5}{69}}. \qquad \text{[repl. } y$$

This process consists in replacing the two given equations by two new equations got from the last two remainders, the one free from x, and the other having x only in the first degree.

So, to find x, y from the pair of quadratic equations

$$x^2 + 2y^2 = 3xy,\ \ 16x - 12y = 5xy;$$

then $\because$ the resultant of these two equations is

$5y^4 - 14y^3 + 8y^2 = 0$, whose linear factors are

$y,\ y,\ (5y-4)(y-2)$,

$$\therefore\ y = 0,\ 0,\ \tfrac{4}{5},\ 2,$$

and $$x = 0,\ 0,\ \tfrac{4}{5},\ 4.$$

The locus of the equation $x^2 + 2y^2 - 3xy = 0$, *i.e.*, of the equation $(x-2y)(x-y) = 0$, consists of the two straight lines whose equations are $x - 2y = 0$, $x - y = 0$.

The four values of y give the ordinates of the four points where the hyperbola whose equation is $16x - 12y = 5xy$ cuts these two straight lines, and two of these four intersections coincide at the origin; hence the double solution, $x = 0$, $y = 0$.

NOTE 1. If both the equations be of the form

$$ax^2 + bxy + cy^2 + f = 0,\ \ a'x^2 + b'xy + c'y^2 + f' = 0,$$

the following method may be adopted: subtract f times the second equation from f' times the first, and divide the resulting equation by y^2; solve for $(x:y)$; replace x by a function of y, and solve for y.

More generally: if the two equations consist of two nth degree homogeneous functions equated to constants: eliminate these constants and thus obtain a homogeneous nth degree function equated to zero. Divide this equation by y^n, and solve the nth degree equation involving the ratio $x:y$.

More generally still: if one function be of the mth degree, and the other of the nth degree, divide the l. c. mlt. of m, n by each of them, raise the two functions to the powers shown by the quotients, and put them equal to like powers of the constants, thus making both equations of the same degree. Solve as above.

NOTE 2. The two equations may be written

$$\mathrm{P}_0 + \mathrm{P}_1 + \cdots + \mathrm{P}_m = 0, \quad \mathrm{Q}_0 + \mathrm{Q}_1 + \cdots + \mathrm{Q}_n = 0;$$

wherein P_0, Q_0 are free from x, y; P_1, Q_1 are homogeneous and of the first degree; $\cdots$ P_m, Q_n are homogeneous and of the degrees m, n.

If the given equations be so incomplete that only a few of the expressions $\mathrm{P}_0, \cdots \mathrm{P}_m, \cdots \mathrm{Q}_0, \cdots \mathrm{Q}_n$ be present, it is often best to put vx for y in both equations, eliminate x, and get the values of v, and then of x, y. [Or else put vy for x and eliminate y.]

By thus putting vx for y, the equations become

$$\mathrm{P}_0 + \mathrm{U}_1 \cdot x + \cdots + \mathrm{U}_m \cdot x^m = 0, \quad \mathrm{Q}_0 + \mathrm{V}_1 \cdot x + \cdots + \mathrm{V}_n \cdot x^n = 0,$$

wherein $\mathrm{U}_1, \cdots \mathrm{U}_m, \mathrm{V}_1^m, \cdots \mathrm{V}_n$, are known quantities or functions of v whose degrees $\ngtr 1, 2, \cdots m, \; 1, 2, \cdots n$.

This method is similar in principle to that of Note 1.

E.g., In the example of Note 1, $\mathrm{P}_0 + \mathrm{P}_2 = 0$, $\mathrm{Q}_0 - \mathrm{Q}_2 = 0$, wherein $\mathrm{P}_0, \mathrm{Q}_0, \mathrm{P}_2, \mathrm{Q}_2 \equiv f, f', \; ax^2 + bxy + cy^2, \; a'x^2 + b'xy + c'y^2$; $\mathrm{U}_2, \mathrm{V}_2 \equiv a + bv + cv^2, \; a' + b'v + c'v^2$; and v is found from the quadratic $(cf' - c'f)v^2 + (bf' - b'f)v + af' - a'f = 0$.

NOTE 3. Sometimes the solution of a pair of equations may be simplified by changing the unknown elements: notably by making use of the following relations, connecting the sum of two numbers, their difference, their product, the difference of their squares, the sum of their squares:

half sum	+ half difference	= greater number.
half sum	− half difference	= less number.
product sum	× difference	= difference of squares.
sum of squares	+ twice product	= square of sum.
sum of squares	− twice product	= square of difference.
(half sum)^2	+ (half difference)^2	= half sum of squares.
(half sum)^2	− (half difference)^2	= product.

When each equation is symmetric as to x, y, it is commonly best to take symmetric functions of x, y for the new unknown elements.

When one equation is symmetric as to x, y, and the other as to x, $-y$, it is often best to take $x+y$, $x-y$ for the new unknown elements.

Sometimes equations not originally thus symmetric may be made so by transformation.

E.g., the resultant of the pair of equations

$$3xy - 4x - 4y = 0, \quad x^2 + y^2 + x + y - 26 = 0 \quad \text{is}$$
$$9y^4 - 15y^3 - 242y^2 + 624y - 416 = 0,$$

which is not easily reduced to a quadratic.

But put $(x+y)^2 - 2xy$ for $x^2 + y^2$, and write the equations:

$$3xy - 4(x+y) = 0, \quad (x+y)^2 + (x+y) - 2xy - 26 = 0,$$

then $\quad x + y = 6, \ xy = 8 \ \text{or} \ x + y = -\frac{13}{3}, \ xy = -\frac{52}{9};$

and $\quad$ the four values of x, y are found from these two pairs of equations, each consisting of a simple equation and a quadratic.

So, to find x, y from the pair of equations

$$x - y = \tfrac{1}{4}xy, \quad x^2 + y^2 = \tfrac{5}{2}xy;$$

square the first equation and subtract from the second to find values of xy; join each of these equations with the first equation to find values of x, y.

So, in the equations $x + y = 4$, $x^4 + y^4 = 82$,

put $\quad u + v \equiv x, \ u - v \equiv y;$

then $\quad (u+v) + (u-v) = 4, \quad u = 2;$

and $\quad (u+v)^4 + (u-v)^4 = 82, \quad u^4 + 6u^2v^2 + v^4 = 41,$

$$\therefore v^4 + 24v^2 - 25 = 0,$$
$$\therefore v^2 = 1 \ \text{or} \ -25, \ v = \pm 1 \ \text{or} \ \pm 5i.$$
$$\therefore x = 3, \ 1, \ 2 + 5i, \ 2 - 5i;$$
$$y = 1, \ 3, \ 2 - 5i, \ 2 + 5i.$$

So, to find the five values of x, y from the pair of equations

$$x + y = 4, \quad x^5 + y^5 = 244;$$

then $\because\ (u+v)+(u-v)=4,\ (u+v)^5+(u-v)^5=244,$

$\therefore\ u=2;\quad u^5+10\,u^3v^2+5\,uv^4=122,$

$\therefore\ v^4+8\,v^2=9;$

$\therefore$ the five values of v are $\infty, {}^+1, {}^-1,\ +i\sqrt{3},\ -i\sqrt{3},$

the five values of x are ${}^+\infty, {}^+3, {}^+1,\ 2+i\sqrt{3},\ 2-i\sqrt{3},$

and the five values of y are ${}^-\infty, {}^+1, {}^+3,\ 2-i\sqrt{3},\ 2+i\sqrt{3}.$

NOTE 4. The meaning of these infinite solutions may be interpreted as follows:

Consider the equation $x+y=4$ as the limiting form of an equation $x+by=4$, whose coefficient b gradually approaches unity as a limit: one of the pairs of values of x, y grow larger and larger without bounds, and the solution is either $x={}^+\infty$, $y={}^-\infty$, or $x={}^-\infty$, $y={}^+\infty$, according as b is a little less than unity or a little greater.

NOTE 5. Sometimes the roots of higher equations may be found by the method of division.

E.g., of the pair of equations

$$y(x^2+y^2)=4(x+y)^2,\quad xy=4(x+y),$$

the resultant is $y^4-8y^3=0;$

and $\because$ this function of y is divisible by y^3,

$\therefore$ the equation has three roots 0; it has also one root 8.

But $\because$ the general resultant of a cubic and a quadratic equation is of the sixth degree,

$\therefore$ this resultant has lost its two highest terms, and the equation has two roots ∞;

$\therefore$ the values of y are ∞, ∞, 0, 0, 0, 8,

and the six corresponding values of x, found from the equation $x=4\,y:(y-4)$ are 4, 4, 0, 0, 0, 8.

The geometrical interpretation of these roots is, that of the six points of intersection of the loci that represent the two equations two are at an infinite distance and lie on the line $x=4$, three are at the origin, and one is at the point whose co-ordinates are 8, 8; or, in the language of limits, if one of the curves

change its form slightly, by the gradual change (say) of a single coefficient, and thus approach its present form, then two of the points of intersection recede to an infinite distance, three of them not coincident approach the origin, and one approaches the point 8, 8.

So, of the pair of equations

$$x^3+y(xy-1)=0,\quad y^3-x(xy+1)=0,$$

the resultant, found from the last remainder, is

$$4y^9-4y^5-y=0,$$

and the second last remainder gives

$$(2y^4+1)x-2y^3=0,$$

$\therefore\ y=0$ or $y^4=\frac{1}{2}(1\pm\sqrt{2})$;

i.e., y may have the value 0 or any one of the eight values of $\sqrt[4]{\frac{1}{2}(1\pm\sqrt{2})},\ =\sqrt[4]{(\frac{1}{2}\pm\sqrt{\frac{1}{2}})}$,

and $x,=2y^3:(2y^4+1)$, may have the value 0 or any one of the eight values of $\sqrt[4]{1:(2\pm2\sqrt{2})},=\sqrt[4]{(^-\frac{1}{2}\pm\sqrt{\frac{1}{2}})}$.

NOTE 6. SPECIAL METHODS OF SOLUTION: Many sets of simultaneous equations may be solved by special devices. The examples given below are meant merely as suggestions to the reader. He is advised to try his own ingenuity upon each example before studying the solution here shown; and afterward, to see how far the principle of each solution applies to other examples.

1. To find the values of x, y from the pair of equations

$$x^4+x^2y^2+y^4=133,\ (1)\quad x^2-xy+y^2=7:\ (2)$$

then $x^2+xy+y^2=19$, (3) [div. (1) by (2)

and $x^2+y^2=13$; (4) [add (2), (3)

$\therefore\ xy=6$, (5) [sub. (4) from (3)

$x^2+2xy+y^2=25$, (6) [add (3), (5)

and $x^2-2xy+y^2=1$; (7) [sub. (5) from (2)

$\therefore\ x+y=\pm5,\quad x-y=\pm1$;

$\therefore\ x,y={}^+3,{}^+2;\ {}^-3,{}^-2;\ {}^+2,{}^+3;\ {}^-2,{}^-3.$

2. If $x^4-x^2+y^4-y^2=84,\quad x^2+x^2y^2+y^2=49$:

then $(x^2+y^2)^2-2x^2y^2-(x^2+y^2)=84$,

$(x^2+y^2)+x^2y^2=49$.

Put $u \equiv x^2 + y^2,\ v \equiv x^2 y^2$; (1)

then $\because\ u^2 - u - 2v = 84,\quad u + v = 49$; (2)

$\therefore\ u^2 + u = 182$; [add (1) and twice (2)

$\therefore\ u = 13$ or -14.

(1) Put $x^2 + y^2 = 13$;

then $\because\ x^2y^2 = 36,\quad x^2 + 36x^{-2} = 13$;

$\therefore\ x^2 = 9$ or 4, $y^2 = 4$ or 9;

$\therefore\ x = \pm 3$ or ± 2, $y = \pm 2$ or ± 3;

i.e., $x, y = {}^+3, {}^+2;\ {}^+3, {}^-2;\ {}^-3, {}^+2;\ {}^-3, {}^-2;\ {}^+2, {}^+3;\ {}^+2, {}^-3;\ {}^-2, {}^+3;\ {}^-2, {}^-3$;
eight pairs of roots.

(2) Put $x^2 + y^2 = -14$;

then $x^2y^2 = 63,\quad x^2 + 63x^{-2} = -14$;

$\therefore\ x^2 = -7 \pm i\sqrt{14},\qquad y^2 = -7 \mp i\sqrt{14}$:

$\therefore\ x = \pm\sqrt{(-7 \pm i\sqrt{14})},\quad y = \pm\sqrt{(-7 \mp i\sqrt{14})}$;
eight pairs of roots.

The plats of these two equations intersect in only eight real points; the other eight points of intersection are imaginary.

3. If $\sqrt{(x+y)} + \sqrt{(x-y)} = \sqrt{a},\ \sqrt{(x^2+y^2)} + \sqrt{(x^2-y^2)} = b$:

then $2x + 2\sqrt{(x^2-y^2)} = a,\qquad 2x^2 + 2\sqrt{(x^4-y^4)} = b^2$, [sqr.

and $x^2 - y^2 = \frac{1}{4}a^2 - ax + x^2,\qquad x^4 - y^4 = \frac{1}{4}b^4 - b^2x^2 + x^4$,
[div. by 2, transp., sqr.

i.e., $y^2 = ax - \frac{1}{4}a^2,\qquad y^4 = b^2x^2 - \frac{1}{4}b^4$;

$\therefore\ b^2x^2 - \frac{1}{4}b^4 = (ax - \frac{1}{4}a^2)^2$;

i.e., $(a^2 - b^2)x^2 - \frac{1}{2}a^3 \cdot x + (\frac{1}{16}a^4 + \frac{1}{4}b^4) = 0$;

$\therefore\ x = [a^3 \pm b(a^2 - 2b^2)] : 4(a^2 - b^2)$,

$y^2 = (ax - \frac{1}{4}a^2) = ab \cdot [ab \pm (a^2 - 2b^2)] : 4(a^2 - b^2)$;

$y = \pm\sqrt{[ab \cdot (ab \pm a^2 \mp 2b^2) : 4(a^2 - b^2)]}$.

4. If $x(x+y+z) = 18,\quad y(x+y+z) = 12,\quad z(x+y+z) = 6$:

then $(x+y+z)(x+y+z) = 36$, [add

$\therefore\ x + y + z = \pm 6$,

$\therefore\ x = \pm 3,\quad y = \pm 2,\quad z = \pm 1$. [div.

5. If $xyz = a^2(y+z) = b^2(z+x) = c^2(x+y)$:

then $\quad 1 \quad = a^2\left(\dfrac{1}{zx}+\dfrac{1}{xy}\right) = b^2\left(\dfrac{1}{xy}+\dfrac{1}{yz}\right) = c^2\left(\dfrac{1}{yz}+\dfrac{1}{zx}\right).$

Put $u, v, w \equiv \dfrac{1}{yz}, \dfrac{1}{zx}, \dfrac{1}{xy}$;

then $\because 1 = a^2(v+w) = b^2(w+u) = c^2(u+v)$,

$$\therefore v+w = \frac{1}{a^2},\quad w+u = \frac{1}{b^2},\quad u+v = \frac{1}{c^2},$$

$$\therefore u+v+w = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),$$

$$\therefore u = \frac{1}{2}\left(-\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right),\quad v = \frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}\right),$$

$$w = \frac{1}{2}\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right);$$

$$\therefore x^2, = \frac{1}{yz} : \left(\frac{1}{zx}\cdot\frac{1}{xy}\right), = u : (w\cdot v)$$

$$= 2\left(-\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) : \left(\frac{1}{a^2}-\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right).$$

So for y^2, z^2.

6. If $yz + zx + xy = 26, \qquad (1)$

$yz(y+z) + zx(z+x) + xy(x+y) = 162, \qquad (2)$

$yz(y^2+z^2) + zx(z^2+x^2) + xy(x^2+y^2) = 538: \qquad (3)$

Write (2) in the form

$$yz(x+y+z) + zx(x+y+z) + xy(x+y+z) - 3xyz = 162,$$

then $\quad (yz+zx+xy)(x+y+z) - 3xyz = 162,$

and $\quad 26(x+y+z) - 3xyz = 162. \qquad (4) \quad$ [(1)

So write (3) in the form

$$(yz+zx+xy)(x^2+y^2+z^2) - xyz(x+y+z) = 538;$$

then $\quad 26[(x+y+z)^2 - 2(yz+zx+xy)]$

$\qquad\qquad - xyz(x+y+z) = 538. \qquad (5)$

Put $u, v \equiv x+y+z,\ xyz$;

then $\quad 26u - 3v = 162,\quad 26u^2 - uv = 1890, \qquad$ [sub. in (4, 5)

$$\therefore u = 9 \text{ or } -12\tfrac{3}{26},\quad v = 24 \text{ or } -159,$$

i.e., $\quad x+y+z = 9$ or $-12\tfrac{3}{26}$, $xyz = 24$ or -159;

but $\because\ x(y+z)+yz=26,$ [(1)

$\therefore\ x(9-x)+24x^{-1}=26,$ [use first vals. for u, v

i.e., $x^3-9x^2+26x-24=0,$

or $(x-2)(x-3)(x-4)=0;$ [factor

$\therefore\ x=2,\ 3,\ 4;$ [th. 2 cr. 1

and $y=2,\ 3,\ 4,\quad z=2,\ 3,\ 4.$ [symmetry

These roots may be grouped in six different ways:

$$\begin{aligned} x&=2,\ 2,\ 3,\ 3,\ 4,\ 4;\\ y&=3,\ 4,\ 4,\ 2,\ 2,\ 3;\\ z&=4,\ 3,\ 2,\ 4,\ 3,\ 2.\end{aligned}$$

So, for the sets of values from the other values of u, v.

7. If $y^2+yz+z^2=7,$ (1)

$z^2+zx+x^2=13,$ (2)

$x^2+xy+y^2=3:$ (3)

then $(x-y)(x+y+z)=6,\quad (z-y)(x+y+z)=10,$

[sub. (1, 3) fr. (2)

$\therefore\ x-y:z-y=3:5,$

i.e., $5x-5y=3z-3y$ and $2y+3z=5x;$ (4)

but $\because\ z^2+x(z+x)=13,$ [(2)

$\therefore\ z^2+\frac{1}{5}(2y+3z)\cdot\frac{1}{5}(2y+8z)=13,$ [sub. for x fr. (4)

i.e., $4y^2+22yz+49z^2=325,$ (5)

$\therefore\ y=\pm 2,\ \pm\dfrac{1}{\sqrt{19}};\quad z=\mp 3,\ \pm\dfrac{11}{\sqrt{19}};\quad x=\mp 1,\ \pm\dfrac{7}{\sqrt{19}}.$

8. If $x^2-yz=a$ (1), $y^2-zx=b$ (2), $z^2-xy=c$ (3):

From the square of one subtract the product of the other two;

then $x(x^3+y^3+z^3-3xyz)=a^2-bc\equiv A,$

$y(x^3+y^3+z^3-3xyz)=b^2-ca\equiv B,$

and $z(x^3+y^3+z^3-3xyz)=c^2-ab\equiv C.$

$\therefore\ Ay=Bx,\quad Az=Cx;$

$\therefore\ (A^2-BC)x^2=aA^2;$ [sub. for y, z in (1)

$$\therefore\ x=\frac{\sqrt{a}\cdot A}{\sqrt{(A^2-BC)}}=\frac{a^2-bc}{\sqrt{(a^3+b^3+c^3-3abc)}}.$$

So $$y=\frac{b^2-ca}{\sqrt{(a^3+b^3+c^3-3abc)}},\quad z=\frac{c^2-ab}{\sqrt{(a^3+b^3+c^3-3abc)}}.$$

§ 16. SPECIAL PROBLEMS INVOLVING QUADRATICS.

For definition of special problems and for the method of putting such problems into equation and of discussing the solutions, see § 10. These methods are best shown by examples.

1. Two farmers at a fair each spent \$1100. A bought 50 sheep and 12 cows; B bought 50 more sheep than cows; and the sum spent by the two together in the purchase of sheep was half the joint expenditure. What was the price of cows and what the price of sheep, and how many sheep and how many cows did B buy?

Let u, v, x, $y \equiv$ the price of the sheep, the price of cows, the number of sheep, and the number of cows bought by B;

then $\because$ $50u+12v=1100$, (1) $\quad xu+yv=1100$, (2)

$50u+xu=1100$, (3) $\quad x=y+50$; (4)

$\therefore$ $(x-50)u+(y-12)v=0$, $\quad xu-12v=0$, [sub. (1) fr. (2) and fr. (3)

$\therefore$ $yu+(y-12)v=0$, $\quad (y+50)u-12v=0$, [sub. for x fr. (4)

$\therefore$ $[12y+(y+50)(y-12)]u=0$. [elim. v

Put $12y+(y+50)(y-12)=0$; [th. 2 cr. 1

then $y=10$ or -60.

Reject the root -60 as absurd;

then $y=10$; $x, =y+50, =60$;

$\therefore$ $50u+12v=1100$, $\quad 60u+10v=1100$, [sub. for x, y in (1, 2)

$\therefore$ $u=10$, $\quad v=50$;

i.e., B bought 60 sheep at \$10, and 10 cows at \$50.

2. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel be increased one yard, the fore-wheel will make only 4 revolutions more than the hind-wheel in the same space; find the circumference of each wheel.

Let x, $y \equiv$ circumferences of fore-wheel and hind-wheel;

then $\dfrac{120}{x}-6=\dfrac{120}{y}$, $\dfrac{120}{x+1}-4=\dfrac{120}{y+1}$; and $x=4$, $y=5$.

§ 17. BINOMIAL EQUATIONS.

PROB. 14. TO SOLVE AN EQUATION OF THE FORM $x^n = a^n$.

Transpose a^n, *giving* $x^n - a^n = 0$; *factor* $x^n - a^n$, *and put the factors severally equal to* 0; *solve the equations thus formed.*

The roots of these new equations are the roots sought. [th. 4 cr. 1

1. To solve the equation $x^2 = a^2$:

then $\because\ x^2 - a^2 = (x-a)\cdot(x+a)$,

$\therefore\ x - a = 0$, or $x + a = 0$,

$\therefore\ x = a$, or $-a$.

2. To solve the equation $x^2 = -a^2$:

then $\because\ x^2 + a^2 = (x - a\sqrt{-1})\cdot(x + a\sqrt{-1})$, [VIII. th. 2, df. imag.

$\therefore\ x = a\sqrt{-1}$, or $-a\sqrt{-1}$,

i.e., $x = ai$, or $-ai$.

3. To solve the equation $x^3 = a^3$:

then $\because\ x^3 - a^3 = (x-a)\cdot(x^2 + ax + a^2)$,

$\therefore\ x - a = 0$, or $x^2 + ax + a^2 = 0$,

$\therefore\ x = a$, or $\frac{1}{2}a(-1 \pm i\sqrt{3})$. [sol. quad.

4. To solve the equation $x^3 = -a^3$:

then $\because$ the equation $x^3 = -a^3$ gives $x^3 = a^3$ if $-x$ replace x,

$\therefore\ x = -a$, or $\frac{1}{2}a(1 \pm i\sqrt{3})$.

5. To solve the equation $x^4 = a^4$:

then $\because\ x^4 - a^4 = (x^2 - a^2)\cdot(x^2 + a^2)$,

$\therefore\ x = a,\ -a,\ ai,\ -ai$. [1, 2

6. To solve the equation $x^4 = -a^4$:

then $\because\ x^4 + 2a^2x^2 + a^4 = 2a^2x^2$, [add $2a^2x^2$

i.e., $(x^2 + a^2)^2 - 2a^2x^2 = 0$;

$\therefore\ (x^2 + a^2 - ax\sqrt{2})\cdot(x^2 + a^2 + ax\sqrt{2}) = 0$,

$\therefore\ x = \frac{1}{2}a(-\sqrt{2} \pm i\sqrt{2}),\ \frac{1}{2}a(\sqrt{2} \pm i\sqrt{2})$. [sol. quad.

7. To solve the equation $x^5 = a^5$:

then $\because$ $x^5 - a^5 = (x-a)\cdot(x^4 + x^3a + x^2a^2 + xa^3 + a^4)$,

$\therefore$ $x = a$,

and $x^4 + x^3a + x^2a^2 + xa^3 + a^4 = 0$;

$\therefore$ $x^2 + ax + a^2 + a^3x^{-1} + a^4x^{-2} = 0$, [div. by x^2

$\therefore$ $(x^2 + a^4x^{-2}) + a(x + a^2x^{-1}) + a^2 = 0$,

$\therefore$ $(x + a^2x^{-1})^2 + a(x + a^2x^{-1}) - 2a^2 + a^2 = 0$,

$\therefore$ $x + a^2x^{-1} = \frac{1}{2}a(-1 \pm \sqrt{5})$,

$\therefore$ $x = a$, $\frac{1}{4}a[(\sqrt{5} - 1) \pm i\sqrt{(10 + 2\sqrt{5})}]$,

$-\frac{1}{4}a[(\sqrt{5} + 1) \pm i\sqrt{(10 - 2\sqrt{5})}]$. [sol. quad.

8. To solve the equation $x^5 = -a^5$:

then $\because$ the equation $x^5 = -a^5$ gives $x^5 = a^5$ if $-x$ replace x,

$\therefore$ $x = -a$, $-\frac{1}{4}a[(\sqrt{5} - 1) \pm i\sqrt{(10 + 2\sqrt{5})}]$,

$\frac{1}{4}a[(\sqrt{5} + 1) \pm i\sqrt{(10 - 2\sqrt{5})}]$.

9. To solve the equation $x^6 = a^6$:

then $\because$ $x^6 - a^6 = (x^3 - a^3)\cdot(x^3 + a^3)$,

$\therefore$ $x = a$, $\frac{1}{2}a(-1 \pm i\sqrt{3})$, $-a$, $\frac{1}{2}a(1 \pm i\sqrt{3})$. [3, 4

10. To solve the equation $x^6 = -a^6$:

then $\because$ the equation $x^6 = -a^6$ gives $x^6 = a^6$ if ix replace x,

$\therefore$ $x = ai$, $\frac{1}{2}a(-i \pm \sqrt{3})$, $-ai$, $\frac{1}{2}a(i \pm \sqrt{3})$.

And so on for other roots.

NOTE. Another method of solution is shown in X. Prob. 1, in finding the nth roots of a^n and of $-a^n$.

§ 18. LOGARITHMIC AND EXPONENTIAL EQUATIONS.

THE methods of solving such equations are set forth in IX. Probs. 3, 8.

E.g., to find x from the equation $15^{2x} + 6\cdot 15^x = 51975$:

then $\because$ $15^x = 225$ or -231, [sol. quad.

$\therefore$ $x = \log 225 : \log 15 = 2.3522 : 1.1761 = 2$;

but of the equation $15^x = -231$ no solution is possible.

§ 19. EXAMPLES.

§ 1.

1. From the following statements pick out the sufficient conditions, the necessary conditions, the equivalent statements, the associated statements, the incompatible statements, the independent statements:
$x<6,\ x=3,\ x<5,\ x>4,\ 2x=6,\ x^2=9,\ x=3,\ x \nless 3.$

2. Give examples in which one statement is a $\left\{\begin{matrix}\text{necessary,}\\ \text{sufficient,}\end{matrix}\right.$ but not a $\left\{\begin{matrix}\text{sufficient,}\\ \text{necessary,}\end{matrix}\right.$ condition of another.

3. Show that if one statement $\left\{\begin{matrix}\text{be}\\ \text{be not}\end{matrix}\right.$ a $\left\{\begin{matrix}\text{necessary}\\ \text{sufficient}\end{matrix}\right.$ condition of another, then the latter $\left\{\begin{matrix}\text{is}\\ \text{is not}\end{matrix}\right.$ a $\left\{\begin{matrix}\text{sufficient}\\ \text{necessary}\end{matrix}\right.$ condition of the former; give examples of these four cases.

§ 5, PROB. 1.

... 24. Solve the equations:

4. $12-5x=13-x;\quad 1-5x=7x+3;\quad 6x-5(3x-7)-21=0.$
5. $a-2x=x-b;\quad m-nx=px+q;\quad ax-b(x-1)-c=0.$
6. $(x+1)(x-1)=x(x-2);\quad (x+4)(x-2)=(x-9)(x-3).$
7. $(x+a)(x-b)=(x-c)(x+d);\quad (x-m)(x+n)=x(x-q).$
8. $ax-m-2\{bx-n-3[cx-p-4(dx-q)]\}=0.$
9. $2x-\frac{4-x}{3}=\frac{11}{3};\quad \frac{7x+4}{9}-\frac{5-x}{3}=\frac{22}{3}-\frac{x}{2}-\frac{8-7x}{6}.$
10. $\frac{1}{a}(x-1)-\frac{1}{b}(x-2)=\frac{2}{b};\quad \frac{1}{m}(5x-6)-\frac{1}{n}(x-1)=x-2.$
11. $\frac{1}{3}(x+10)-\frac{3}{5}(3x-4)+\frac{1}{6}(3x-2)(2x-3)=x^2-\frac{8}{15}.$
12. $\frac{1}{x}-\frac{3}{2x}+\frac{5}{7x}=\frac{3}{28};\quad \frac{1}{x+1}+\frac{2}{x+2}=\frac{3}{x+3}.$
13. $\frac{2x-3}{3x+4}=\frac{6x+5}{9x-10};\quad \frac{x-a}{2x-b}=\frac{3x-c}{6x-d}.$
14. $\frac{1}{x-3}-\frac{1}{x-4}=\frac{1}{x-5}-\frac{1}{x-6};\quad \frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}.$

15. $\frac{1}{2}\left(x-\frac{a}{3}\right)+\frac{1}{3}\left(x-\frac{a}{4}\right)+\frac{1}{4}\left(x-\frac{a}{5}\right)=0$; $\frac{a}{x-a}+\frac{b}{x-b}=\frac{a+b}{x-c}$.

16. $(x-a)^3+(x-b)^3+(x-c)^3=3(x-a)(x-b)(x-c)$.

17. $(x^2-3x+4)^{\frac{1}{2}}=x-3$; $[2(1-x)(3-2x)]^{\frac{1}{2}}=2x+1$.

18. $(8-4x)^{\frac{1}{2}}+(13-4x)^{\frac{1}{2}}=5$; $2x+\sqrt{[4x^2+\sqrt{(1-4x)}]}=1$.

19. $18:\sqrt{(2x+3)}=\sqrt{(2x-3)}+\sqrt{(2x+3)}$.

20. $3\sqrt{(x-\frac{5}{9})}+7\sqrt{(x+\frac{8}{49})}=10\sqrt{(x+\frac{8}{100})}$.

21. $\sqrt{(3x+1)}-\sqrt{[2-x+2\sqrt{(1-x)}]}=1$.

22. $\sqrt[3]{(\sqrt{3}+x\sqrt{7})}+\sqrt[3]{(\sqrt{3}-x\sqrt{7})}=\sqrt[6]{12}$.

23. $\frac{1+x-\sqrt{(2x+x^2)}}{1+x+\sqrt{(2x+x^2)}}=a\frac{\sqrt{(2+x)}-\sqrt{x}}{\sqrt{(2+x)}+\sqrt{x}}$.

24. $\sqrt{(a+x)}+\sqrt{(a-x)}=b[\sqrt{(a+x)}-\sqrt{(a-x)}]$.

§§ 6, 7, PROBS. 2, 3, 4.

...28. From the following pairs of equations eliminate one of the variables by the first method, and solve the equations:

25. $8x+3y=14$, $5y=10$; $3x-8y=7$, $3\frac{1}{2}x=5$.

26. $15x+2y=17$, $9x-4y=5$.

27. $210x+42y+93=0$, $22x+14y+7=0$.

28. $\frac{3}{8}y-\frac{1}{5}x+24=0$, $\frac{2}{7}y+\frac{1}{3}x+11=0$.

...31. So, by the second method:

29. $x+y=9$, $x-y=1$; $5x+3y=8$, $7x-3y=4$.

30. $3x+y=16$, $3y+x=8$; $3y=5x$, $16y=27x-1$.

31. $\frac{1}{3}\left(\frac{x}{4}-\frac{y}{5}+\frac{1}{6}\right)=\frac{1}{4}(x-y)$, $\frac{1}{2}\left(\frac{y}{4}-\frac{x}{5}+\frac{1}{3}\right)=\frac{1}{4}(x+y)$.

...35. So, by the third method:

32. $11x-3y=0$, $x-y=-16$; $x=\frac{2}{5}y$, $x-\frac{1}{2}y=\frac{2}{8}$.

33. $x-y=\frac{5}{6}$, $x+1=\frac{3}{2}(y+x)$.

34. $\frac{x+y}{x-2y}=3$, $\frac{x-3y}{6}+\frac{5y-x}{9}=\frac{1}{2}$.

35. $\frac{1}{15}(80+3x)=18\frac{1}{3}-\frac{1}{7}(4x+3y-8)$,
$10y+\frac{1}{5}(6x-35)=55+10x$.

…39. Eliminate x by the fourth method from:

36. $4x - 7y = 9,\ \ 16x^2 - 49y^2 = 207.$

37. $x^2 + xy = 7,\ \ x + y = 4;\ \ x^2 + 3xy = 10,\ \ y^2 + 2xy = 5.$

38. $x^2 + xy + y^2 = 5,\ \ 2x^2 + 3xy + 4y^2 = 11.$

39. $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1,\ \ x^2 + y^2 = 2.$

§ 7, PROB. 5.

…48. Solve the systems of equations:

40. $3x - 4y + 5z = 4,\ \ 8x - y - z = 6,\ \ 7x - 5y - 3z = -1.$

41. $x + y + z = 6,\ \ x - y + z = 2,\ \ x + y - z = 0.$

42. $x - 2y - 5z = 20,\ \ 3x - 5y - 3z = 22,\ \ -8x + 11y + 9z = -57.$

43. $2(x+1) - 3(y-1) + z - 2 = 2,$
$2(x-1) + 4(y+1) - 5(z-1) = 3,$
$3(2x+2) - 2(y-1) + 3(z+1) = 29.$

44. $\frac{1}{x} - \frac{1}{y} + \frac{1}{z} = \frac{11}{6},\ \ \frac{2}{x} - \frac{3}{y} + \frac{4}{3z} = \frac{71}{18},\ \ \frac{3}{x} + \frac{4}{y} - \frac{1}{7z} = \frac{20}{21}.$

45. $3\tfrac{1}{3}x + 5\tfrac{1}{2}y - 1\tfrac{1}{11}z = 51,\qquad y^2 + z^2 = 2a^2 + \tfrac{1}{2}x^2,\ \ cy + bz = a,$
$2\tfrac{1}{4}x + 3\tfrac{1}{3}y - 1\tfrac{1}{2}z = 23\tfrac{3}{4},\qquad z^2 + x^2 = 2b^2 + \tfrac{1}{2}y^2,\ \ az + cx = b,$
$1\tfrac{1}{5}x + 2\tfrac{1}{7}y + \tfrac{3}{4}z = 31\tfrac{127}{140};\ \ x^2 + y^2 = 2c^2 + \tfrac{1}{2}z^2;\ \ bx + ay = c.$

46. $x + 2y + 3z + 4u = 20,\ \ x + 2y + 3z - 4u = 12,$
$x + 2y - 3z + 4u = 8,\ \ x - 2y + 3z + 4u = 8.$

47. $3x - 4y + 3z + 3v - 6u = 11,$
$3x - 5y + 2z - 4u = 11,\ \ 10y - 3z + 3u - 2v = 2,$
$5z + 4u + 2v - 2x = 3,\ \ 6u - 3v + 4x - 2y = 6.$

48. $5x - 2(y + z + v) = -1,\ \ -12y + 3(z + v + x) = 3,$
$4z - 3(v + x + y) = 2,\ \ 8v - (x + y + z) = -2.$

Denote $x + y + z + v$ by s; from these equations respectively express x, y, z, v in terms of s; substitute these values in any one of the equations; solve for s; and thence find x, y, z, v.

§ 8.

… 52. Plat the lines that represent the equations:

49. $x = 0,\ \ y = 0,\ \ x = 4,\ \ x = -4,\ \ y = 4,\ \ y = -4,\ \ x = \pm a,$
$y = \pm b,$ [a, b, lines of any known length

50. $y = x,\ \ y = -x,\ \ y = 2x,\ \ y = -3x,\ \ 2y = 3x,\ \ 3y = -2x,$
$ly = mx,\ \ ly = -mx,$ [l, m any two given numbers

51. $y=x+2,\ y=-x+2,\ y=x-2,\ y=-x-2,$
$y=mx+c,$ [m any given number, c any given line

52. $2x+3y+5=0,\ 3x-2y-5=0,\ lx+my+c=0,$
[l, m any given numbers, c any given line

53. Find the lengths of the intercepts upon the axes of the lines whose equations are given in Exs. 52–55.

54. Find the co-ordinates of the points of intersection of the lines whose equations are
$y=x,\ y=-x;\ y=x+2,\ y=x-2;$
$2x+3y+5=0,\ 3x-2y-5=0;\ y=mx+c,$
$y=-mx+c,$ [m any given number, c any given line

55. Find the co-ordinates of the vertices of the triangles bounded by the lines that represent the equations:
$2x+3y+5=0,\ 3x-2y-5=0,\ x=5;$
$ax+by+c=0,\ a'x+b'y+c'=0,\ a''x+b''y+c''=0.$

56. Find the co-ordinates of the vertices of the parallelograms bounded by the lines that represent the equations:
$2x+3y+5=0,\ 2x+3y-5=0,$
$3x-2y+5=0,\ 3x-2y-5=0.$

§ 9.

57. By aid of Bezout's method solve examples 42–51.

§ 10, PROB. 6.

58. Find two numbers, such that their sum is 27; and that, if four times the first be added to three times the second, the sum is 93.

59. Find two numbers, such that twice the first and three times the second together make 18; and if double the second be taken from five times the first, 7 remains.

60. A flagstaff is sunk in the ground one-sixth part of its height, the flag occupies 6 feet, and the remainder of the staff is three-quarters of its whole length; what is the height of the flagstaff?

61. The diameter of a five-franc piece is 37 millimeters, and of a two-franc piece is 27 millimeters; thirty pieces laid in contact in a straight line measure one meter; how many of each kind are there?

62. Find three numbers such that the sum of the first and second is 15; of the first and third, 16; and of the second and third, 17.

63. The sum of the three digits of which a number consists is 9; the first digit is one-eighth of the number consisting of the last two, and the last digit is likewise one-eighth of the number consisting of the first two.

64. At an examination there were 17 candidates, of whom some were passed, some conditioned, and the rest rejected; if one less had been rejected, and one less conditioned, the number of those passed would have been twice those rejected, and five times those conditioned; how many of each class were there?

65. There are three candidates at an election, at which it is necessary that at least one more than half the entire number of electors should vote for the successful candidate; A fails to obtain an absolute majority, although he has 20 votes more than B; but supposing that C, whose votes are only three-tenths of B's, had withdrawn, and that one-fourth of his supporters voted for A, then A would have been barely successful; how many voted for each candidate?

66. A gentleman left a sum of money to be divided among four servants; the first was to have half as much as the other three together, the second one-third as much as the other three, and the third one-fourth as much as the other three; the first, moreover, was to have $70 more than the last; how much should each get?

67. A father divides his estate among his children as follows: to the first a dollars and the nth part of the remainder; to the second, $2a$ dollars and the nth part of the remainder; to the third, $3a$ dollars and the nth part of the remainder; and so on. It results that in the entire division of the estate each child receives the same amount. Find the value of the estate, the number of children, and the amount each one receives.

68. In a company of a persons each man gave m dollars to the poor, and each woman n dollars; the whole amount collected was ka dollars; how many men were there, and how many women?

Show that, if $m > n$, then $m > k > n$.

Show that the example is possible only when $(m-k)a$, $(k-n)a$ are multiples of $m-n$ and have the same sign as $m-n$.

69. Upon a horizontal straight line let O be a fixed point, let A lie a units to the left of O, and B, b units to the right of O; find on this line a third point X such that if M be the middle of BX, then AO is one-third of AM.

Show that if $4a > b$, X lies to the left of O; if $4a = b$, X coincides with O; if $4a < b$, X lies to the right of O.

70. A reservoir holding V gallons is filled in h hours by p pipes, all of the same size, and by the rain falling uniformly on a roof of s square yards. Another reservoir holding V′ gallons is filled in h' hours by p' pipes of the same size as the others, and the rain falling uniformly and with the same intensity as before upon a roof of s' square yards. Find x, the inflow per hour of each pipe, and y, the rainfall per hour on each square yard of roof.

Explain the meaning of the problem if for particular values of the constants either x or y or both of them be negative.

71. Two circles of radii r, r' lie in the same plane and have their centres d units apart; find the point where the exterior common tangents cut the line that joins the centers.

Show by the formula that if the smaller circle grows while the larger stands fast, the point recedes farther and farther away; that when the growing circle is of the same size as the other, that point has gone to infinity (does not exist); and that when the growing circle passes the other, the point reappears upon the other side at infinity, and creeps back toward the circles.

72. Find the four terms of a proportion that exceed by the same number the four numbers a, b, c, d.

Discuss the solution when (1) $ad = bc$, (2) $a + d = b + c$.

73. Given the series $a+b$, $ap+bq$, ap^2+bq^2, ap^3+bq^3, ap^4+bq^4, $\cdots$, to find two numbers x, y, such that each term of this series after the second can be got by multiplying the one before it by x, and the one before that by y, and adding the products.

74. Given the series $a+b+c$, $ap+bq+cr$, $ap^2+bq^2+cr^2$, $\cdots$, to find three numbers x, y, z, such that each term of this series after the third may be found by multiplying the one before it by x, the one before that by y, and the one before that by z, and adding the products.

75. A laborer receives a dollars a day when he works, and forfeits b dollars a day when idle. At the end of m days he receives k dollars; how many days does he work, and how many is he idle?

What relation exists between the given elements if his forfeits just cancel his earnings? if his forfeits exceed his earnings?

Give numerical illustrations.

76. A father is now a times as old as his son; k years hence he will be b times as old; what are their ages now?

Give numerical values to a, b, k, and interpret the results.

Show that: $k>0$ if $a>b$; $k=0$ if $a=b$; $k<0$ if $a<b$.

77. The sum of two numbers is a, and the difference of their squares is k^2; what are the numbers?

Interpret the results if (1) $k^2>a^2$; (2) $k^2=a^2$; (3) $k^2<a^2$.

78. The difference of two numbers is a, and the difference of their squares is k^2; what are the numbers?

Interpret the results if (1) $k^2>a^2$; (2) $k^2=a^2$; (3) $k^2<a^2$.

79. If to the numerator of a certain simple fraction a be added, the result is $\frac{c}{d}$, and if to the denominator a' be added, the result is $\frac{c'}{d'}$; what is the original fraction?

Show what relations must exist between the constants so that $\frac{c}{d}$, $\frac{c'}{d'}$ shall be simple fractions and in their lowest terms.

Give numerical illustrations.

80. In a certain two-digit number the second digit is a times the first, and if b be added to the number, the digits are reversed.

Show that a may not exceed 9, nor be negative; and show when a may be fractional.

Show that b is a multiple of 9; and show what bounds b lies between for different values of a.

81. A yacht steams up a river m miles and down the river n miles, in h hours; again she steams up the river m' miles and down the river n' miles, in h' hours; what is the rate of the yacht in still water, and what the current of the river, the speed of the yacht and the current of the river being uniform?

Give numerical illustrations and discuss all possible cases.

82. A dealer has three kinds of tea, worth 25 cents, 50 cents, and one dollar, a pound; how shall he mix them by even pounds so that 50 pounds of the mixed tea shall be worth $30?

83. Two vases A, B hold V, V′ gallons, and are each filled with a mixture of wine and water, A in the proportion $m : n$, B in the proportion $m' : n'$. Two other vases C, D are of equal size and hold less than A or B; C is filled from A, and D from B at the same time; C is emptied into B, and D into A; and then the proportion of wine to water is the same in A, B; of what size are the vases C, D?

84. Of two ingots the first has a parts gold, b parts silver, the second has a' parts gold, b' parts silver; in what proportion shall they be combined so that the product shall have c parts gold, d parts silver?

Between what bounds do c, d lie?

85. If A can do a units of work in a' days, B, b units in b' days, C, c units in c' days; in how many days can they do $a + b + c$ units, all working together?

What is the value of c' if the whole work be finished in h days?

Give numerical illustrations.

86. To do a certain work A needs m times as long as B and C, B n times as long as C and A, C p times as long as A and B; find the relation between m, n, p.

87. Two right triangles have their right angles coincident, and the sides about that angle extend along a horizontal line a, a' yards, and along a vertical line b, b' yards; find how far the point of meeting of the two hypothenuses lies to the right and above the vertex of the right angle.

Discuss all the possible cases.

88. The points A, B, C, ⋯ lie on a straight line, at distances a, b, c ⋯ from a fixed point O upon the line; find a point X on this line such that its distance from any point M on the line shall be the average of the distances of A, B, C ⋯ from M.

Show that the result is independent of M.

89. A reservoir is filled by pipes A, B in c hours, by pipes B, C in a hours, by pipes C, A in b hours; in what time is it filled by each pipe running alone? and by all three running together?

Give numerical illustrations and discuss all possible cases.

Show what relations must exist between a, b, c so that no water flows through either A, or B, or C, or any two of them; and what relations must exist so that one or two of the pipes shall give an outflow.

90. A reservoir holding m gallons is filled by two pipes, A, B, running a, b gallons an hour, and emptied by two pipes, C, D, running c, d gallons an hour. What is the relation between a, b, c, d, so that, with all the pipes running, the reservoir shall be filled in h hours? that it shall be emptied in k hours?

So with one pipe running in and two out? or two in and one out?

Make a general formula to involve all the pipes, counting the outflow as negative inflow.

Give numerical values to the letters, and interpret the results in special cases.

91. A hound pursues a fox, and makes a leaps while the fox makes b leaps; but c hound-leaps equal d fox-leaps.

The fox has a start of k leaps; when will the hound overtake the fox?

Give numerical values to the letters, and interpret the results.

What is the relation between a, b, c, d, so that the hound shall never catch the fox? what the relation so that the fox is running away from the hound? what the relation so that the hound will catch the fox?

92. Two couriers, A, B, are at M, N, d miles apart, and going forward at a, b miles an hour; when are they together?

Consider the following cases: (a) when they move towards each other; (b) when away from each other; (c) when in the same direction, A behind B, and going (1) faster than B, (2) slower, (3) at the same rate.

Interpret the several results, and illustrate by giving numerical values to a, b, d.

93. Three couriers, A, B, C, are all upon the same straight road, and going at a, b, c miles an hour. They are now at the points M, N, P, distant M to N, h miles, N to P, k miles.

Find when A will be midway between B, C; B midway between C, A; C midway between A, B.

Show what special relations must exist between a, b, c, so that they may be all together.

Take distances to the right, and time forward, positive; distances to the left, and time past, negative. Find the general formulæ; and interpret the several results when different numerical values are given to the letters.

94. If the hour and minute hands of a clock be together at 12 noon, at what other times between noon and midnight will they be together? at what times will they be opposite to each other? at what times will they be at right angles to each other?

Apply the fact that the minute hand gains 55 minute spaces in 60 minutes.

95. If the hour, minute, and second hands of a clock all turn on the same pivot, and if they be together at 12 noon. when will they next be together? at what times will the second hand be midway between the other two? at what times will the minute hand be midway between the other two? at what times will the hour hand be midway between the other two? at what times will they divide the clock face into three equal spaces? at what times will they form a T with either hand as the body, and the other two as the head of the T?

96. If three planets, A, B, C, circle about the sun in the same direction, and with orbits in the same plane, in a, b, c years, and if they be now in conjunction (on the same side of the sun, and all in a straight line with it), when will they be again in conjunction? when will A, B be in conjunction, and C in opposition? when will B, C be in conjunction, and A in opposition? when will C, A be in conjunction, and B in opposition? when will they so stand that the arc AB subtends an angle θ at the sun, the arc BC an angle θ', the arc CA an angle θ''?

97. Three boys, A, B, C, starting together, run round a circular m-yard track, at a, b, c yards a second; find general formulæ for the times of: their conjunction, the conjunction of two of them and opposition of the third, the division of the track into arcs d, e, f, such that $d+e+f=m$; first when all run in the same direction, second when two run in the same direction and one in the opposite direction.

Show that the last formulæ are identical with the first if the speed of the one be called negative.

Show what relations must exist between a, b, c, that the runners may never again be all together.

§ 11, PROB. 7.

...101. Find the values of x from the equations:

98. $(x^2+1)(x^2+2)=(x^2+6)(x^2-1)$.

99. $\frac{1}{2}(x^2-\frac{1}{6}a^2)-\frac{1}{3}(x^2-\frac{1}{8}a^2)+\frac{1}{4}(x^2-\frac{1}{10}a^2)=0$.

100. $\frac{1}{5}(3x^2-7)+\frac{1}{9}(25-4x^2)=\frac{1}{3}(5x^2-14)$.

101. $2(2x^2-5)^{-1}+(x^2-3)^{-1}=6(3x^2-1)^{-1}$.

§ 11. PROB. 8.

··· 117. Solve both by completing the square and by factoring:

102. $x^2-8x+15=0,\quad x^2+10x=-24,\quad x^2-5x+4=0$.

103. $6x^2-19x+10=0,\quad 7x^2-3x=160,\quad 110x^2-21x+1=0$.

104. $(5x-3)^2-7=44x+5,\quad (3x-5)(2x-5)=(x+3)(x-1)$.

105. $\frac{5}{7}x^2+\frac{7}{5}x+\frac{73}{140}=0,\quad (x-2)^{-1}-2(x+2)^{-1}=\frac{3}{5}$.

106. $\dfrac{3x-2}{2x-5}-\dfrac{2x-5}{3x-2}=\dfrac{8}{3},\quad \dfrac{x+3}{x+2}+\dfrac{x-3}{x-2}=\dfrac{2x-3}{x-1}$.

107. $\dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}=\dfrac{a}{b}+\dfrac{b}{a},\quad \dfrac{x+a}{x-a}+\dfrac{x+b}{x-b}+\dfrac{x+c}{x-c}=3$.

108. $x^2-(5+3i)x+\frac{1}{4}(11+13i)=0,\ x^2-(4+3i)x+(7+5i)=0$.

109. $3x+2\sqrt{x}-1=0,\quad x^{\frac{1}{n}}-13x^{\frac{1}{2n}}=14$.

110. $x^4-14x^2+40=0,\quad x^{\frac{1}{3}}+\frac{5}{2}x^{-\frac{1}{3}}=3\frac{1}{4},\quad \sqrt{2x}-7x=-52$.

111. $x+5-\sqrt{(x+5)}=6,\quad \sqrt[n]{x}+\sqrt[n]{x^{-1}}=2\frac{1}{2}$.

112. $\sqrt{(2x+7)}+\sqrt{(3x-18)}=\sqrt{(7x+1)}$.

113. $x+\sqrt{x}+\sqrt{(x+2)}+\sqrt{(x^2+2x)}=a$.

114. $x^2+3=2\sqrt{(x^2-2x+2)}+2x,\quad \sqrt{(x^2-2x+9)}-\frac{1}{2}x^2=3-x$.

115. $3x^2+15x-2\sqrt{(x^2+5x+1)}=2$.

116. $x^2-2\sqrt{(3x^2-2ax+4)}=\frac{2}{3}a(x+\frac{1}{2}a+1)$.

117. $nx^3+x+n+1=0,\quad x^3+x^2-4x-4=0$.

118. Form the quadratic equations whose pairs of roots are:
$2,3$; $1,-4$; $3\pm 2i$; $-1\pm i$; $\pm 3+2i$; $4+5i,\ 1+2i$.

119. Form equations by putting equal 0 the quadratic functions:
$2x^2+x-6$; $6x^2-19x+15$; $x^2-2mx+m^2-n^2$;
$x^2-(m+n)x+(m+p)(n-p)$; $(x-a)^2-b^2$;
$x^2-5(1+i)x+13i,\quad x^2+(7+5i)x+6+17i$.
Solve, and by aid of the roots factor the functions.

120. If α and β be the roots of the equation $x^2-px+q=0$, find the value of $\alpha\beta^{-1}+\beta\alpha^{-1}$ and of $\alpha^3+\beta^3$.

121. What value of c gives the equation $5x^2+3x+c=0$ equal roots?

122. Eliminate x from the two equations

$$ax^2+bx+c=0, \quad a'x^2+b'x+c'=0.$$

123. Show that the quadratic function ax^2+bx+c may be written in the forms $\frac{1}{4a}\{(2ax+b)^2-(b^2-4ac)\}$ and $\frac{1}{4a}\{2ax+b+\sqrt{(b^2-4ac)}\}\{2ax+b-\sqrt{(b^2-4ac)}\}$; hence derive the condition for real and unequal, for equal, and for imaginary factors. By this method factor the function $3x^2+5x+2$, and find for what values of x the function vanishes.

§ 12, PROB. 9.

124. Plat the quadratic functions; hence find the real values of x, if any, that make these functions vanish:

$$x^2-4x+3,\ x^2-4x+3\tfrac{1}{2},\ x^2-4x+4,\ x^2-4x+4\tfrac{1}{2},$$
$$x^2+x+6,\ -x^2-x-6,\ -3x^2-10x+13.$$

§ 13, PROB. 10.

125. Find five convergents to the roots of the equations:

$$x^2+x-6=0,\ x^2-3x+2=0,\ x^2-6x+9=0,$$
$$3x^2+4x=7,\ 4x^2-3x=10,\ 5x^2-10x=20.$$

§ 14, PROB. 11.

126. Find the maximum or minimum values of the functions

$$x^2-4x+3,\ 10+4x-x^2,\ x^2-6x+9,\ -x^2+6x-9;$$

and the corresponding values of x.

127. From the plat of the functions in the examples of § 12, state which of them have maximum values and which minimum, and find these values.

Show that each of these functions has a $\left\{\begin{matrix}\text{maximum}\\ \text{minimum}\end{matrix}\right.$ value if the vertex of the corresponding parabola be $\left\{\begin{matrix}\text{upwards,}\\ \text{downwards,}\end{matrix}\right.$ *i.e.*, if the coefficient of x^2 be $\left\{\begin{matrix}\text{positive.}\\ \text{negative.}\end{matrix}\right.$

128. Show that $\dfrac{x^2-3x+1}{x-3}$ has no value between 1 and 5.

129. Find the maximum value of $(x+a)(x-b):x^2$.

130. Show that $a(a+x):(a-x)$ can have any value.

131. Find the maximum or minimum values of

$$\frac{x^2+2x+11}{x^2+4x+10};\quad \frac{x^2-x+1}{x^2+x-1};\quad \frac{a+x}{a-x}+\frac{a-x}{a+x}.$$

To the last apply the principle that if the product of two variables be constant, their sum is a minimum when they are equal.

132. Prove that the quotient $(x+a):(x^2+bx+c^2)$ always lies between two fixed finite bounds if $a^2+c^2>ab$ and $b^2<4c^2$; that there are two bounds between which it cannot lie if $a^2+c^2>ab$ and $b^2>4c^2$; and that it may take all values if $a^2+c^2<ab$.

133. Find what value of x will make a maximum the product:

$$(x^2+2x+1)\cdot(7-x^2-2x);\quad (x^2-25):(25-x^2).$$

Apply the principle that if the sum of two variables be constant, their product is a maximum when they are equal.

134. Find the sides of the maximum rectangle that can be inscribed in a given circle.

§ 15, PROB. 12.

... 168. Find the values of x, y from the pairs of equations:

135. $x+y=7,\ \ x^2+2y^2=34;\quad x-y=12,\ \ x^2+y^2=74.$

136. $x+y=a,\ \ xy=b^2;\quad x-y=a,\ \ xy=b^2.$

137. $3x-5y=2,\ \ xy=1;\quad x+y=100,\ \ xy=2400.$

138. $x+y=a,\ \ x^2+y^2=b^2;\quad x^{\frac{1}{2}}+y^{-\frac{1}{2}}=4,\ \ x-y^{-1}=8.$

139. $x+y=4,\ \ x^{-1}+y^{-1}=1;\quad 2x+3y=37,\ \ x^{-1}+y^{-1}=\frac{14}{45}.$

140. $x+y=2,\ \ x^2-2xy-y^2=1;\quad x+y=18,\ \ x^3+y^3=4914.$

141. $x+y=72,\ \ \sqrt[3]{x}+\sqrt[3]{y}=6;\quad x^2y^{-1}+y^2x^{-1}=9,\ \ x^{-1}+y^{-1}=\frac{3}{4}.$

§ 15, PROB. 13.

142. $4x^2+7y^2=148,\ \ 3x^2-y^2=11;\quad x+y=x^2,\ \ 3y-x=y^2.$

143. $x^2+y^2=\frac{5}{2}xy,\ \ x-y=\frac{1}{4}xy;\quad x^2+xy=6,\ \ x^2+y=5.$

NOTE 1.

144. $x^2+xy+2y^2=74,\ \ 2x^2+2xy+y^2=73.$

145. $x^2+y^2=a^2,\ \ xy=b^2;\quad x^2-y^2=a^2,\ \ xy=b^2.$

146. $x^2+3xy=54,\ \ xy+4y^2=115;\quad x^3+y^3=9,\ \ xy=2.$

147. $x^2+xy+4y^2=6,\ \ 3x^2+8y^2=14;\quad x^2+y^2=1,\ \ x^3+y^3=1.$

NOTE 2.

148. $x^2-3xy+2y^2=0,\ \ x^2+y^2=x^3-y^3;\quad x^2-y^2=8,\ \ x^3-y^3=26.$

149. $x^2+xy-6y^2=0,\ \ 2x+3y+x^2+3xy+2y^2=19.$

150. $8x^3+3x^2y+xy^2=18,\ \ 2x^3+5x^2y+3xy^2=24.$

NOTE 3.

151. $x+y=5, \quad x^4+y^4=97; \quad x-y=3, \quad x^5-y^5=3093.$

152. $x^4+y^4=14x^2y^2, \quad x+y=9; \quad x^{\frac{1}{2}}+y^{\frac{1}{4}}=1, \quad x^3+y=17.$

153. $x^2+y^2=7+xy, \quad x^3+y^3=6xy-1.$

NOTE 5.

154. $x^4-x^2+y^4-y^2=84, \quad x^2+x^2y^2+y^2=49.$

155. $xy(x+y)=30, \quad x^3+y^3=35.$

NOTE 6.

156. $4(x+y)=3xy, \quad x+y+x^2+y^2=26.$

157. $x^2(x+y)=80, \quad x^2(2x-3y)=80.$

158. $x^4+x^2y^2+y^4=133, \quad x^2-xy+y^2=7.$

159. $x^2+y^2-(x+y)=a, \quad x^4+y^4+x+y-2(x^3+y^3)=b.$

160. $x+y+\sqrt{xy}=14, \quad x^2+y^2+xy=84.$

161. $x^2+y=4x, \ y^2+x=4y; \quad x^3+xy^2=10, \ y^3+x^2y=5.$

162. $x^3+y^3+3x+3y=378, \ x^3+y^3-3x-3y=324;$

$x^2=ax+by, \ y^2=ay+bx; \ bx+ay=ab, \ bx+ay=4xy.$

163. $10x^2+15xy=3ab-2a^2, \ 10y^2+15xy=3ab-2b^2;$

164. $6x^2-3x-4y=25, \ x^2+2x-3y=18;$

165. $xy+6x+7y=50, \ 3xy+2x+5y=72.$

166. $x+y=10, \ \sqrt{xy^{-1}}+\sqrt{yx^{-1}}=\frac{5}{2};$

167. $\sqrt{(x^2+y^2)}+\sqrt{(x^2-y^2)}=2y, \quad x^4-y^4=a^4.$

168. $8x^{\frac{1}{3}}-y^{\frac{1}{2}}=14, \ x^{\frac{2}{3}}y^{\frac{3}{2}}=2y^2.$

... 172. Find the values of x, y, z from the sets of equations:

169. $yz=bc, \ bx+ay=ab, \ cx+az=ac.$

170. $x+y+z=x^{-1}+y^{-1}+z^{-1}=\frac{7}{2}, \quad xyz=1;$

171. $xy=a(x+y), \ xz=b(x+z), \ yz=c(y+z).$

172. $x+y+z=6, \ 4x+y=2z, \ x^2+y^2+z^2=14.$

§ 16.

173. A boat-crew rows $3\frac{1}{2}$ miles down a river and back again in an hour and 40 minutes; if the river have a current of two miles an hour, at what rate does the crew row?

174. A number is composed of two digits; the first exceeds the second by unity, but the number itself falls short of the sum of the squares of its digits by 26; what is the number?

175. A number is composed of two digits; the first exceeds the second by 2; the sum of the squares of the given number and of the number got by reversing the digits is 4034; what is the number?

176. Find the lengths of the segments of a line a, if m times the square of one be equal to n times the rectangle under the whole line and the other.

177. The driving-wheels of a locomotive are 2 feet longer in diameter than the running-wheels; the running-wheels make 140 turns more than the driving-wheels in a mile; what are the diameters? [ratio circum. : diam. = 22 : 7

178. A set off from London to York, and B at the same time from York to London, and they traveled uniformly; A reached York 4 hours, and B reached London 9 hours, after they met; in what time did each make the journey?

179. A broker bought a number of hundred-dollar railway shares at a certain rate discount for \$7500, and afterwards, at the same rate premium, he sold them all but 60 for \$5000; how many did he buy, and what did he give a share?

180. Divide a line 3 feet long into two parts such that the circle standing on one segment as diameter shall be equal to the square standing on the other.

181. The number 563 in the decimal scale is less than the same number in a higher scale by 232; what is the radix of the higher scale?

182. What is the price of eggs when two more in a shilling's worth lowers the price one penny a dozen?

183. There are two numbers whose product is the difference of their squares, and the sum of whose squares is the difference of their cubes; what are the numbers?

184. The sum of the squares of the numerator and denominator of a fraction is 389, and the difference of the fraction and its reciprocal is $\frac{189}{170}$; find the fraction.

185. Find two numbers such that their sum, their product, and the sum of their squares shall be equal to each other.

186. Find two numbers whose product is p, and the difference of whose cubes is m times the cube of their difference.

187. Find a fraction the product of whose numerator and denominator is 180, and such that if its numerator and denominator be each increased by 10, its value is doubled.

188. A rectangular space, whose length and breadth are 42 and 78 feet, is surrounded by a ditch 5 feet deep, and capable of holding 220 tons of water; what is the breadth of the ditch, counting 6 tons of water for a cubic fathom?

189. There is a fraction such that if the numerator be increased and the denominator diminished by 2, the reciprocal of the fraction is the result; but if the numerator be diminished and the denominator increased by 2, the result is less than the reciprocal by $1\frac{1}{15}$; what is the fraction?

Solve the same problem in general terms, replacing 2 and $1\frac{1}{15}$ by a and b.

190. Two boys set off from the right angle of a right-triangular field, running in opposite directions, with speeds in the ratio of 13 : 11; they first meet at the middle point of the hypothenuse, and again at a point 30 yards distant from the starting-point; find the lengths of the three sides.

191. Two cubical vessels together hold 407 cubic inches; when one vessel is placed on the other, the total height is 11 inches; find the contents of each.

192. A number consists of two digits, the difference of whose squares is 40, and if it be multiplied by the number consisting of the same digits taken in reverse order, the product is 2701; find the number.

193. A vessel can be filled with water by two pipes; by one of these pipes alone the vessel would be filled 2 hours sooner than by the other; and the vessel can be filled by both pipes together in $1\frac{7}{8}$ hours; find the time that each pipe alone would take to fill the vessel.

194. A vessel is to be filled with water by two pipes; the first pipe is kept open during three-fifths of the time which the second would take to fill the vessel; then the first pipe is closed and the second is opened; had both pipes been kept open, the vessel would have been filled 6 hours sooner, and the first pipe would have brought in two-thirds of the water which the second pipe did bring; how long would each pipe alone take to fill the vessel?

195. A number consists of three digits; the first is to the second as the second is to the third; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it, the digits are reversed; what is the number?

196. The diagonal of a box is 125 inches, the area of the lid is 4500 square inches, and the sum of three conterminous edges is 215 inches; find the lengths of these edges.

197. One side of a room is 5 feet longer than the other side, and 1000 square feet of paper is needed to cover its walls; if it were 3 feet higher, the same paper would be needed for 3 only of its walls, the bare wall being one of its longer sides; what are the dimensions of the room?

§ 17. PROB. 14.

198. Solve the binominal equations: $x^3-1=0$, $x^3+1=0$, $x^6=-8$, $x^8=16$, $x^{10}+1=0$, $x^{12}+1=0$, $x^{12}-1=0$.

199. Find the square root to three decimal places of:
$5+12i$, $12+5i$, $161-240i$, $13+7i$, $7+13i$.

200. Prove that the n roots of the equation $x^n=a+bi$, are all given by the expression

$$\sqrt[n]{r}\cdot\left(\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right),$$

wherein r is the tensor and θ the versorial angle of the number $a+bi$, and k has any n consecutive values in the series of natural numbers between $^-\infty$ and $^+\infty$.

§ 18.

201. Find the value of x from the exponential equations:
$2^x=8$, $2^{x+3}=8^{x-3}$, $3^{\frac{x+2}{x-2}}=27^{\frac{x+1}{x-1}}$, $9^x=3$, $8^{3x+1}=2$.

202. By aid of the table of logarithms find x from the equations:
$10^x=3$, $4^x=10$, $.3^x=.8$, $3^{2x+3}=100^{x-6}$, $15^{3x+4}=27^{4x-5}$.

203. Solve the equations: $3^{2x}-7.3^x=18$, $2^{6x}-5.2^{3x}+6=0$, $2^{x+1}+4^x=80$, $4.3^{2x+1}-5.3^{x+2}=12$.

204. If $ak^{2rx+m}+2bk^{rx+n}+c=0$, prove that
$x=[\log\{-bk^n\pm\sqrt{(b^2k^{2n}-ack^m)}\}-\log(ak^m)]:r\log k$.

XII. SERIES.

☞ For definition of *series*, see I. § 12. The first and last terms of a series are its *extremes;* the other terms are its *means.*

§ 1. ARITHMETIC PROGRESSION.

An Arithmetic Progression is a series such that each term after the first is formed by adding a constant to the next preceding term. The constant added is the *common difference.*

The abbreviations are: a for first term, l for last term, d for common difference, n for number of terms, s for sum of all the terms.

When d is $\left\{\begin{matrix}\text{positive,}\\ \text{negative,}\end{matrix}\right.$ the series is $\left\{\begin{matrix}\text{an } \textit{ascending}\\ \text{a } \textit{descending}\end{matrix}\right.$ progression.

E.g., 1, 3, 5, 7, 9, is an ascending series,
wherein $d = {}^{+}2$, $a = 1$, $l = 9$, $n = 5$, $s = 25$.

So, 9, 7, 5, 3, 1, $^{-}1$, $^{-}3$, is a descending progression,
wherein $d = {}^{-}2$, $a = 9$, $l = {}^{-}3$, $n = 7$, $s = 21$.

Theor. 1. *In an arithmetic progression*

1] $$l = a + (n-1)d.$$

For $\because$ $a+d$, $a+2d$, $a+3d$, $\cdots$, $a+(k-1)d$
are the 2d, 3d, 4th, $\cdots$ kth terms, [df.

$\therefore$ $a+(n-1)d = l$, the last of a series of n terms. Q.E.D.

Cor. *In an arithmetic progression*

2] $$a = l - (n-1)d,$$

3] $$d = \frac{l-a}{n-1},$$

4] $$n = \frac{l-a}{d} + 1.$$

The reader may prove, solving formula 1 in turn for a, d, n.

THEOR. 2. *In an arithmetic progression*

5] $s = \frac{1}{2}n(a+l)$.

For $\because\ s = a + (a+d) + (a+2d) + \cdots + (l-d) + l$, n terms,

and $s = l + (l-d) + (l-2d) + \cdots + (a+d) + a$, n terms,

$\therefore\ 2s = (a+l) + (a+l) + (a+l) + \cdots + (a+l)$, n times,

$= n \cdot (a+l)$.

$\therefore\ s = \frac{1}{2}n(a+l)$. Q.E.D.

COR. 1. *In an arithmetic progression*

6] $a = \frac{2s}{n} - l$,

7] $l = \frac{2s}{n} - a$,

8] $n = \frac{2s}{a+l}$.

The reader may prove, solving formula 5 in turn for a, l. n.

COR. 2. *In an arithmetic progression*

9] $l = \frac{2s + n(n-1)d}{2n}$,

10] $d = \frac{2(nl-s)}{n(n-1)}$,

11] $n = \frac{d + 2l \pm \sqrt{[(2l+d)^2 - 8ds]}}{2d}$,

12] $s = \frac{1}{2}n[2l - (n-1)d]$,

13] $a = \frac{2s - n(n-1)d}{2n}$,

14] $d = \frac{2(s-an)}{n(n-1)}$,

15] $n = \frac{d - 2a \pm \sqrt{[(2a-d)^2 + 8ds]}}{2d}$,

16] $s = \frac{1}{2}n[2a + (n-1)d]$,

17] $a = \frac{1}{2}\{d \pm \sqrt{[(2l+d)^2 - 8ds]}\}$,

18] $l = \frac{1}{2}\{-d \pm \sqrt{[(2a-d)^2 + 8ds]}\}$,

19] $d = \frac{(l+a)(l-a)}{2s - (l+a)}$,

20] $s = \frac{(l+a)(l-a+d)}{2d}$.

The reader may prove formulæ 9–12, combining 1, 5 so as to eliminate a, then solving in turn for l, d, n, s; formulæ 13–16, by eliminating l, then solving for a, d, n, s; formulæ 17–20, by eliminating n, then solving for a, l, d, s.

NOTE 1. The formulæ involving a may be got from those involving l, and *vice versa*, by symmetry, writing a in place of l, l in place of a, and $-d$ in place of $+d$; and thus seven of the fourteen formulæ 1, 2, 6, 7, 9–18 may be written directly from the other seven; for if any arithmetic progression be reversed, then a becomes l, l becomes a, and d becomes $-d$.

NOTE 2. Formulæ 11, 15 give two values for n. If either of these values be negative or fractional, it may be rejected as inconsistent with the conditions of the problem. [XI. pr. 6 nt. 3

PROB. 1. TO INSERT m ARITHMETIC MEANS BETWEEN a, l.

Divide the remainder, $l-a$, *by* $m+1$ *for the common difference; and to* a *add one, two, three,* ⋯ *times this difference.*

E.g., to insert 5 means between 12 and 48:

then $\because$ $(48-12):(5+1)=6$, the common difference,

$\therefore$ the series sought is 12, 18, 24, 30, 36, 42, 48.

NOTE. By aid of this problem, from every arithmetic progression a new arithmetic progression may be formed by inserting the same number of arithmetic means between every two consecutive terms; and the common difference of this new progression is the quotient of the common difference of the other divided by one more than the number of terms so inserted.

So from any arithmetic progression a new progression may be formed by taking equidistant terms;

E.g., if two means be inserted between two consecutive terms:

then 6, 12, 18, 24, 30, ⋯

becomes 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, ⋯

and if of this new progression the first, fifth, ninth, ⋯ terms be taken, a third progression is formed,

6, 14, 22, 40, ⋯

whose common difference is $4\cdot 2, =8$.

The reader may prove formulæ 29–32, combining formulæ 21, 25 so as to eliminate a, then solving in turn for l, r, n, s; formulæ 33–36 by eliminating r, then solving for a, l, n, s; formulæ 37–40 by eliminating n, then solving for a, l, r, s. He will observe that formulæ 30, 33, 34 have no general solutions. In numerical equations their solution is always possible.

NOTE. The formulæ involving a may be got from those involving l, and *vice versa*, by symmetry, writing a in place of l, l in place of a, and r^{-1} in place of r^{+1}; and thus seven of the fourteen formulæ 21, 22, 25–34, 37, 38 may be written directly from the other seven; for if any geometric progression be reversed, then a becomes l, l becomes a, and r^{+1} becomes r^{-1}.

PROB. 2. TO INSERT m GEOMETRIC MEANS BETWEEN a, l.

Take the $(m+1)$th *root of the quotient* $l:a$ *for the common ratio; and multiply* a *by the first, second ⋯ powers of this ratio.*

E.g., To insert three means between 3 and 48:

then $\because\ \sqrt[4]{(48:3)} = 2$, the common ratio,

$\therefore$ the series sought is 3, 6, 12, 24, 48.

NOTE. By aid of this problem, from every geometric progression a new geometric progression may be formed by inserting the same number of geometric means between every two consecutive terms; and the common ratio of this new progression is that root of the common ratio of the other whose index is one more than the number of means so inserted.

So, from any geometric progression a new progression may be formed by taking equidistant terms.

E.g., if two means be inserted between two consecutive terms,

then 3, 6, 12, 24, ⋯

becomes $3,\ 3\sqrt[3]{2},\ 3\sqrt[3]{4},\ 6,\ 6\sqrt[3]{2},\ 6\sqrt[3]{4},\ 12,\ 12\sqrt[3]{2},\ 12\sqrt[3]{4},\ 24, \cdots$,

and if of this new progression the first, fifth, ninth, ⋯ terms be taken a third progression is formed

$$3,\quad 6\sqrt[3]{2},\quad 12\sqrt[3]{4},\ \cdots$$

whose common ratio is the fourth power of either of the three values of $\sqrt[3]{2}$.

§ 3. HARMONIC PROGRESSION.

A HARMONIC PROGRESSION is a series such that any three consecutive terms being taken, the ratio of the first to the third equals the ratio of the excess of the first over the second to the excess of the second over the third.

E.g., if p, q, r, be any three consecutive terms of a harmonic progression, then $p : r = p - q : q - r$.

THEOR. 5. *If a series of numbers be in harmonic progression their reciprocals are in arithmetic progression; and conversely.*

Let p, q, r be any three consecutive terms of a harmonic progression;

then will $r^{-1} - q^{-1} = q^{-1} - p^{-1}$.

For $\because p : r = p - q : q - r$, [df.

$\therefore pq - pr = pr - qr$, [II. th. 6

$\therefore r^{-1} - q^{-1} = q^{-1} - p^{-1}$; Q.E.D. [div. by pqr

So for the converse.

PROB. 3. TO INSERT m HARMONIC MEANS BETWEEN TWO EXTREMES, a, l.

Find m *arithmetic means between* a^{-1} *and* l^{-1}, *and take their reciprocals.*

E.g., to insert two harmonic means between 12 and 48;

then $\because \frac{1}{12} - \frac{1}{48} = \frac{3}{48}$, and $\frac{3}{48} : 3 = \frac{1}{48}$,

$\therefore$ the arithmetic progression is $\frac{1}{12}, \frac{1}{16}, \frac{1}{24}, \frac{1}{48}$, [pr. 1

and the harmonic progression is 12, 16, 24, 48, [th. 5

wherein $12 : 24 = 12 - 16 : 16 - 24$, $16 : 48 = 16 - 24 : 24 - 48$.

NOTE. The analogies and relations of the three progressions appear below: If p, q, r be three numbers

in $\left\{\begin{array}{l}\text{arithmetic}\\ \text{geometric progression,}\\ \text{harmonic}\end{array}\right.$ then $p - q : q - r = \left\{\begin{array}{l}p : p;\\ p : q;\\ p : r;\end{array}\right.$

and the $\left\{\begin{array}{l}\text{arithmetic}\\ \text{geometric mean of } p, r \text{ is}\\ \text{harmonic}\end{array}\right.$ $\left\{\begin{array}{l}\frac{1}{2}(p + r).\\ \sqrt{pr}.\\ 2pr : (p + r).\end{array}\right.$

So, the geometric mean of p, r is the geometric mean of the arithmetic and harmonic means of p, r.

THEOR. 6. *If four numbers,* p, q, r, s, *be so related that* p − q, p − r, p − s *form a harmonic progression, then:*

(*a*) q − r, q − s, q − p *likewise form a harmonic progression; and so do* r − s, r − p, r − q; and s − p, s − q, s − r.

(*b*) *The relations between* p, q, r, s *shown in* (*a*) *hold true also:*

1. *Among any four numbers,* n+p, n+q, n+r, n+s, *whose differences equal the differences of* p, q, r, s;

2. *Among any equimultiples of* p, q, r, s; *or of their reciprocals;*

3. *Among* $\frac{ap+b}{cp+d}, \frac{aq+b}{cq+d}, \frac{ar+b}{cr+d}, \frac{as+b}{cs+d}$,

wherein a, b, c, d *are any numbers.*

(*a*) ∵ the condition that $\frac{1}{p-q}+\frac{1}{p-s}=\frac{2}{p-r}$ [th. 5

is that $(p+r)\cdot(q+s)=2pr+2qs$; [free fr. fracts., red.

and ∵ the condition that $\frac{1}{q-r}+\frac{1}{q-p}=\frac{2}{q-s}$

is that $(q+s)\cdot(r+p)=2qs+2rp$, [ch. p, q, r, s to q, r, s, p

i.e., that $(p+r)\cdot(q+s)=2pr+2qs$, as above,

∴ when $p-q$, $p-r$, $p-s$ form a harmonic progression, so do $q-r$, $q-s$, $q-p$. Q.E.D.

So do $r-s$, $r-p$, $r-q$; and $s-p$, $s-q$, $s-r$.

(*b*) ∵ relation (*a*) involves p, q, r, s only by their differences,

∴ it holds for any numbers, $n+p$, $n+q$, $n+r$, $n+s$. Q.E.D.

2. ∵ equation $(p+r)\cdot(q+s)=2pr+2qs$ is not changed when for $p, \cdots s$ are put $np, \cdots ns$; or $n:p \cdots n:s$;

∴ the equation is true for these, if for $p, \cdots s$. Q.E.D. [above

3. ∵ $\frac{ap+b}{cp+d}\equiv\frac{a}{c}+\frac{bc-ad}{c^2p+cd}$, $\frac{aq+b}{cq+d}\equiv\cdots$, $\cdots$, [division

and ∵ when relation (*a*) holds for $p, \cdots s$ it holds for $c^2p, \cdots c^2s$, [2

i.e., for c^2p+cd, $\cdots$ c^2s+cd, [1

i.e., for $\frac{bc-ad}{c^2p+cd}$, $\cdots$ $\frac{bc-ad}{c^2s+cd}$, [2

i.e., for $\frac{a}{c}+\frac{bc-ad}{c^2p+cd}$, $\cdots \frac{a}{c}+\frac{bc-ad}{c^2s+cd}$; [1

∴ it holds for $\frac{ap+b}{cp+d}$, $\cdots \frac{as+b}{cs+d}$, if for $p, \cdots s$. Q.E.D.

§ 4. CONVERGENCE AND DIVERGENCE.

☞ IN this section all series are understood to be infinite, and to be made up of real, positive terms only.

The *sum of a series* is the limit of the sum of its first n terms when n becomes indefinitely great.

The excess of the sum of a series over the sum of its first n terms is its *remainder after* n *terms*.

The abbreviations are: s for the sum of the series, T_n for the nth term, s_n for $T_1 + T_2 + \cdots + T_n$, the sum of the first n terms, and R_n for $T_{n+1} + T_{n+2} + \cdots$, the remainder after n terms.

An infinite series is $\left\{ \begin{matrix} \textit{convergent} \\ \textit{divergent} \end{matrix} \right.$ if $s_n \left\{ \begin{matrix} \doteq \\ \not\doteq \end{matrix} \right.$ a finite limit;

i.e., if $R_n \left\{ \begin{matrix} \doteq \\ \not\doteq \end{matrix} \right. 0$, when $n \doteq \infty$.

The terms of an infinite convergent series grow smaller and smaller, since $R_n, \equiv T_{n+1} + T_{n+2} + \cdots, \doteq 0$; but that this condition is not sufficient appears from an example:

In the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ the terms grow smaller;

but the series is not convergent; for if it be grouped thus:

$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots,$$

then ∵ the sum of no group is less than $\frac{1}{2}$,

and ∵ the series consists of an infinite number of such groups,

∴ $s_n \not\doteq$ a finite limit when $n \doteq \infty$. Q.E.D.

THEOR. 7. *The sum of a convergent series of positive terms is the same in whatever way the terms are arranged or grouped.*

Let $s \equiv T_1 + T_2 + T_3 + \cdots$, any convergent series; let the same series be arranged or grouped in any other way, say $(T_2 + T_1) + (T_4 + T_3) + \cdots$;

and let s_n' be the sum of the first n groups;

then will $s_n' \doteq s$ when $n \doteq \infty$.

For ∵ in s are found all the terms of s_n' and more;

∴ $s_n' < s$, and $\lim s_n' \ngtr s$. [df. limit

So $s_n < \lim s_n'$, and $s, = \lim s_n, \ngtr \lim s_n'$;

∴ $\lim s_n' = s$. Q.E.D.

THEOR. 8. *If the terms of a convergent series be multiplied by any same finite number, the new series thus formed is convergent.*

Let $T_1 + T_2 + T_3 + \cdots + T_n + \cdots$ be any convergent series, and k a constant;

then is the series $kT_1 + kT_2 + kT_3 + \cdots + kT_n + \cdots$ convergent.

For $\because R_n, = T_{n+1} + T_{n+2} + \cdots, \doteq 0$ when $n \doteq \infty$, [hyp.

$\therefore kR_n, = kT_{n+1} + kT_{n+2} + \cdots, \doteq 0$ when $n \doteq \infty$. Q.E.D.

COR. *If the terms of a convergent series be multiplied by any finite numbers not larger than a given finite number, the new series thus formed is convergent.*

THEOR. 9. *If, after a given term, the terms of a series form a decreasing geometric progression, the series is convergent.*

Let $T_1 + T_2 + T_3 + \cdots + T_k + r \cdot T_k + r^2 \cdot T_k + \cdots$ be a series such that the terms after a given term T_k form a geometric progression with r smaller than 1;

then is this series convergent.

For $\because T_1 + \cdots + T_k$ is a finite constant number, S_k,

and $\because T_{k+1} + \cdots = T_{k+1} \cdot (1 + r + r^2 + \cdots)$

$\doteq T_{k+1} : 1 - r$, when $n \doteq \infty$, [th. 4 cr. 2

$\therefore S = S_k + \dfrac{T_{k+1}}{1-r}$, a finite number. Q.E.D.

THEOR. 10. *If one series be convergent, and if the terms of another series be not larger than the corresponding terms of the first series, the second series is convergent.*

Let ${}^+T_1 + {}^+T_2 + {}^+T_3 + \cdots + {}^+T_n + \cdots$ be a convergent series,

and let $T_1' + T_2' + T_3' + \cdots + T_n' + \cdots$ be another series such that

$T_1' \not> T_1, \quad T_2' \not> T_2, \quad T_3' \not> T_3, \quad \cdots, \quad T_n' \not> T_n, \quad \cdots;$

then is the second series convergent.

For $\because T_{n+1}' \not> T_{n+1}, \quad T_{n+2}' \not> T_{n+2}, \quad \cdots,$

$\therefore R_n' \not> R_n,$

and $\because R_n \doteq 0$, [hyp.

$\therefore R_n' \doteq 0$. Q.E.D.

COR. 1. *If one series be divergent, and if the terms of another series be not smaller than the corresponding terms of the first series, the second series is divergent.*

COR. 2. *If one series be convergent, and if in a second series the ratio of each term to the term before it be not larger than the corresponding ratio in the first series, the second series is convergent.*

THEOR. 11. *If, after a given term, the ratio of each term of a series to the term before it be smaller than some fixed number that is itself smaller than unity, the series is convergent.*

Let $T_1 + T_2 + T_3 + \cdots + T_k + \cdots$ be a series such that after a given term T_k the ratios $T_{k+1} : T_k, \cdots$ each $\lessdot h \lessdot 1$;

then is the series convergent.

For form a new series

$T_1 + T_2 + T_3 + \cdots + T_k + T'_{k+1} + \cdots + T_n' \cdots$

identical with first series for the first k terms, and thereafter a geometric progression whose ratio is h;

then $\because$ this second series is convergent, [th. 9

and $\because$ the terms of the first series are not larger than the corresponding terms of the second series, [hyp.

$\therefore$ the first series is convergent. Q.E.D. [th. 10

NOTE 1. It is not sufficient that the ratios $T_{k+1} : T_k, \cdots$ be simply less than a unit.

E.g., the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$. [above

NOTE 2. APPLICATION OF THE THEOREM: To apply this theorem, find the law of the ratio $T_{n+1} : T_n$, which in general is some function of n; then determine whether this ratio r, as n increases, finally becomes and remains smaller than some fixed number h, that is itself smaller than a unit.

If smaller than h, the series is convergent.

If smaller than a unit simply, there is doubt.

If a unit or larger than a unit, the series is divergent.

E.g., given $\frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots$:

then $\because$ the ratio $T_n : T_{n-1} = \frac{1}{n} \doteq 0$ when $n \doteq \infty$,

$\therefore$ the series is convergent.

So, given $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$:

then $\because$ the ratio $\mathrm{T}_n : \mathrm{T}_{n-1} = \dfrac{(n-1)^2}{n^2} = (1-\dfrac{1}{n})^2 \doteq 1$ when $n \doteq \infty$,

$\therefore$ there is doubt.

But, if the series be grouped thus:

$$1+\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\left(\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\right)+\left(\frac{1}{8^2}+\frac{1}{9^2}+\cdots+\frac{1}{15^2}\right)+\cdots,$$

i.e., in groups wherein the denominator of the first term of each group is an integral power of 2;

then $\because$ the several groups are less than $\dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \cdots,$

and $\because$ the series $1+\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\cdots$ is convergent, [th. 9

$\therefore$ the first series is convergent. Q.E.D. [th. 10

So, given $1+\dfrac{1}{2^p}+\dfrac{1}{3^p}+\cdots$:

$$\text{then } \because \mathrm{s} = 1+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)$$

$$+\left(\frac{1}{8^p}+\cdots+\frac{1}{15^p}\right)+\cdots,$$

$$<1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\cdots,$$

a geometric series whose ratio is 2^{1-p}.

$\therefore$ s is convergent when $p>1$. [th. 10

So, the series $1+\dfrac{1}{2(\log 2)^p}+\dfrac{1}{3(\log 3)^p}+\cdots,$

$$1+\frac{1}{2\log 2\,(\log\log 2)^p}+\frac{1}{3\log 3\,(\log\log 3)^p}$$

$+\cdots$, and so on,

are $\left\{\begin{matrix}\text{convergent}\\ \text{divergent}\end{matrix}\right.$ when $p \begin{matrix}>\\ \ngtr\end{matrix} 1$. [group as above

NOTE 3. GENERAL TEST OF CONVERGENCE. The series

$$1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots,\ 1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\cdots, \text{etc.},$$

are each of them $\left\{\begin{matrix}\text{convergent}\\ \text{divergent}\end{matrix}\right.$ when $p \begin{matrix}>\\ \ngtr\end{matrix} 1$. [ex. above

These series, when compared with most other series, furnish a test of their convergence. [th.10, th. 8

It is to be noted that in the divergent series $1+\frac{1}{2}+\frac{1}{3}+\cdots$, the nth term, $\frac{1}{n}$, is an infinitesimal when $n \doteq \infty$. Let this infinitesimal be counted the base [VII. § 4, df. order of infls.]; then $\frac{1}{n^p}$, the nth term of the convergent series $1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$, $[p>1]$ is an infinitesimal of an order higher than the first order by a finite number $p-1$. And, conversely, a series whose terms are infinitesimals of an order, p, finitely higher than the first order, is convergent. But if $p>1$, and $p \doteq 1$ when $n \doteq \infty$, there is doubt, and the series may then be tested by the series $1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\cdots$, and so on,

i.e., from the series $\frac{1}{n}$, $\frac{1}{n\cdot\log n}$, $\frac{1}{n\cdot\log n\cdot\log\log n}$, $\cdots$, a base may generally be chosen for which, when $n \doteq \infty$, the order of the term T_n of the series to be tested is $\left\{\begin{matrix}\text{finitely}\\ \text{not}\end{matrix}\right.$ higher than the first order, and the series is $\left\{\begin{matrix}\text{convergent}\\ \text{divergent}\end{matrix}\right.$.

NOTE 4. BOUNDS OF ERROR: In the summation of most series, only a finite number of terms is used, and only approximations to the true value are found; and it is then important to know between what bounds the error lies. That approximation is s_n and the error is $-\text{R}_n$. [V. § 5 df., n finite.

E.g., in the first example of Note 2,

$$\text{R}_n=\text{T}_{n+1}+\text{T}_{n+2}+\cdots=\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots,$$

and $\because$ the terms of this series are not greater than those of

$$\frac{1}{(n+1)!}+\frac{1}{(n+1)!}\cdot\frac{1}{n+1}+\frac{1}{(n+1)!}\cdot\frac{n}{(n+1)^2}+\cdots,$$

$$\therefore\ \text{R}_n \not> \frac{1}{(n+1)!}:\left(1-\frac{1}{n+1}\right),\ =\frac{1}{n\cdot n!};\qquad\text{[th. 4 cr. 2}$$

i.e., the error lies between 0 and $-\frac{1}{n\cdot n!}$.

In particular, $\text{s}_6 \sim \text{s} < \frac{1}{6\cdot 6!}=\frac{1}{4320}$; $\text{s}_{10}\sim \text{s} < \frac{1}{10\cdot 10!}$.

NOTE 5. SERIES ARRANGED TO POWERS OF A VARIABLE: If a series be arranged to the powers of some variable x, thus

$$A_0 + A_1x + A_2x^2 + \cdots + A_nx^n + \cdots,$$

then the ratio $T_{n+1} : T_n = x(A_n : A_{n-1}) = x : (A_{n-1} : A_n)$,

and the series is $\left\{\begin{array}{l}\text{convergent}\\ \text{in doubt}\\ \text{divergent}\end{array}\right.$ if $x \left\{\begin{array}{l}<\\ =\\ >\end{array}\right. A_{n-1} : A_n$, when $n \doteq \infty$.

E.g., the series $1 + x + 2!x^2 + 3!x^3 + \cdots$ is divergent, however small x may be;

for the ratio $A_{n-1} : A_n = 1 : n \doteq 0$, when $n \doteq \infty$.

So, the series $1 + \dfrac{x}{1} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ is convergent, however large x may be.

So, the series $\dfrac{x}{1} + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots$ is $\left\{\begin{array}{l}\text{convergent}\\ \text{divergent}\end{array}\right.$ if $x \left\{\begin{array}{l}<1;\\ \nless 1;\end{array}\right.$

for the ratio $A_{n-1} : A_n \doteq 1$ when $n \doteq \infty$;

and this series is divergent if $x = 1$.

So, the series $x + 2x^2 + 3x^3 + \cdots$ is $\left\{\begin{array}{l}\text{convergent}\\ \text{divergent}\end{array}\right.$ if $x \left\{\begin{array}{l}<1.\\ \nless 1.\end{array}\right.$

That value of x which leaves the series in doubt,

viz., $^+\lim(A_{n-1} : A_n)$ when $n \doteq \infty$,

is the *radius of convergence* of the series.

E.g., if $r \equiv$ radius of convergence, then in the first of the examples above $r = 0$; in the second, $r = \infty$; in the third and fourth, $r = 1$.

If some of the powers of x be wanting, the general method of Note 2 must be applied.

E.g., the series $1 + \dfrac{x^2}{3} + \dfrac{x^4}{7} + \cdots + \dfrac{x^{2(n-1)}}{2^n - 1}$ is $\left\{\begin{array}{l}\text{convergent}\\ \text{divergent}\end{array}\right.$ if $x \left\{\begin{array}{l}<\\ >\end{array}\right. \sqrt{2}$;

for the ratio $T_{n+1} : T_n = x^2(2^n - 1) : (2^{n+1} - 1)$
$\doteq x^2 : 2$ when $n \doteq \infty$;

and the radius of convergence is $\sqrt{2}$.

§ 5. INDETERMINATE SERIES.

AN infinite series that has different sums when its terms are arranged or grouped in different ways is *indeterminate*.

E.g., the sum of the series $^+1$, $^-1$, $^+1$, $^-1$, $^+1$, ... may be

either $\quad (1-1)+(1-1)+(1-1)+\cdots, =0,$

or $\quad 1+(^-1+1)+(^-1+1)+\cdots, \quad =1.$

Indeterminate series, although not always divergent, are here classed with non-convergent series.

THEOR. 12. *An infinite series that has positive and negative terms that separately form divergent series is indeterminate.*

For take any positive term or group of positive terms for $^+\text{T}_1$, leaving positive terms whose sum $^+\text{R}_1$ is infinite,

and from the negative terms, whose sum is infinite, take enough terms so that their sum $^-\text{T}_2$ is larger than $^+\text{T}_1$, leaving negative terms whose sum $^-\text{R}_2$ is infinite;

and from $^+\text{R}_1$ form $^+\text{T}_3$ larger than $^-\text{T}_2$, leaving $^+\text{R}_3$ infinite;

and from $^-\text{R}_2$ form $^-\text{T}_4$ larger than $^+\text{T}_3$, leaving $^-\text{R}_4$ infinite; and so on;

then $\because$ the new series $^+\text{T}_1$, $^-\text{T}_2$, $^+\text{T}_3$, $^-\text{T}_4$, ...

gives $\quad (^+\text{T}_1+{}^-\text{T}_2)+(^+\text{T}_3+{}^-\text{T}_4)+\cdots, \quad =$ some negative number,

and $\quad {}^+\text{T}_1+(^-\text{T}_2+{}^+\text{T}_3)+(^-\text{T}_4+{}^+\text{T}_5)+\cdots, =$ some positive number,

$\therefore$ the series is indeterminate. Q.E.D.

NOTE 1. This result appears also from this, that the sum of the given series reduces to the difference of the sums of two divergent series and is of the form $\infty - \infty$, an indeterminate expression.

COR. 1. *A series* S *is non-convergent if the series got by making all the terms of* S *positive be divergent.*

For, if the series be divergent when all the terms are made positive, it is either of the form $\infty - \infty$, $a - \infty$, $\infty - a$, when part of the terms are made negative;

i.e., it is either indeterminate or divergent. Q.E.D.

NOTE 2. Manifestly a given series may be reduced to the form $^+\text{T}_1$, $^-\text{T}_2$, $^+\text{T}_3$, $^-\text{T}_4$, $\cdots$ in an infinite number of ways, giving an infinite number of such double values.

NOTE 3. An indeterminate series may sometimes be arranged so as to have terms alternately positive and negative and growing smaller and smaller; and if the terms approach 0, the sum for such arrangement has a single finite value, but for different arrangements different values. If, for a particular arrangement, a series have a single finite value, however grouped, the series is convergent for that arrangement.

E.g., if $s = \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \cdots$ towards $^{\pm}1$;

then the two values of s both lie between 0 and 2;

and $s_n \sim s_{n+1} \doteq 1$ when $n \doteq \infty$.

So, if $s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ towards 0,

then $s < 1$, $\because R_1$ is negative,

$> 1 - \frac{1}{2}$, $\because R_2$ is positive,

$< 1 - \frac{1}{2} + \frac{1}{3}$, $\because R_3$ is negative,

and so on;

and $R_n \doteq 0$ when $n \doteq \infty$;

i.e., $s < 1, \; > \frac{1}{2}, \; < \frac{5}{6}, \; \cdots$.

But, if this series be arranged thus:

$$\left(1 + \frac{1}{3}\right) - \frac{1}{2} + \left(\frac{1}{5} + \frac{1}{7}\right) - \frac{1}{4} + \left(\frac{1}{9} + \frac{1}{11}\right) - \frac{1}{6} + \cdots$$
$$+ \left(\frac{1}{4n-3} + \frac{1}{4n-1}\right) - \frac{1}{2n} \cdots;$$

then $s < 1 + \frac{1}{3}, \; > 1 + \frac{1}{3} - \frac{1}{2}, \; < 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7}, \; \cdots$;

i.e., $s < 1\frac{1}{3}, \quad > \frac{5}{6}, \quad < \cdots$.

The reader may group the positive terms by threes, or by fours, or ..., and the negative terms singly, by twos, or by threes, or ..., at his pleasure, taking care that the terms of his new series be always in descending order of magnitude.

THEOR. 13. *A series* s *is convergent if the series got by making all the terms of* s *positive be convergent.*

For let $^+s' \equiv$ the series of positive terms in s,

and $^-s'' \equiv$ the series of negative terms in s;

then $\because$ $^+s' + ^+s''$ is finite, [hyp.

$\therefore$ $^+s'$, $^+s''$ are both finite,

$\therefore$ $^+s'$, $^+s''$ is the same however its terms are arranged or grouped. [th. 7

Let the terms of s be arranged and grouped in any way,

and let $s_n \equiv$ sum of the first n groups of that arrangement,

and $^+s'_{n'} \equiv$ sum of the n' positive terms of s contained in s_n,

and $^-s''_{n''} \equiv$ sum of the n'' negative terms of s contained in s_n;

then $\because$ $s_n = s'_{n'} - ^+s''_{n''}$,

$\therefore$ $\lim s_n = \lim s'_{n'} - \lim ^+s''_{n''} = s' - ^+s''$.

But $\because$ $^+s'$, $^+s''$ are finite constant numbers, [above

$\therefore$ s, $= ^+s' - ^+s''$, is a finite constant number. Q.E.D.

COR. 1. *If a series* s *be* $\left\{\begin{matrix}\textit{convergent,}\\ \textit{non-convergent,}\end{matrix}\right.$ *the series got by making all the terms of* s *positive is* $\left\{\begin{matrix}\textit{convergent.}\\ \textit{divergent.}\end{matrix}\right.$

COR. 2. *If an indeterminate series be convergent for a particular arrangement* $^+T_1, {}^-T_2, {}^+T_3, \dots {}^\pm T_k, {}^\mp T_{k+1}, \dots$, *the ratio* $T_{k+1} : T_k$ *becomes and remains smaller than unity, but approaches unity as its limit.* [th. 11 nt. 2

For if the ratio $T_{k+1} : T_k$ approach a limit h smaller than unity, the series is convergent and not indeterminate. [th. 13, th. 11 nt. 2

NOTE. If indeterminate series be classed with divergent series as above, then, in the light of theors. 12, 13, it appears that theors. 7–11, with their notes and corollaries, apply to series with negative terms, and that those theorems are general for all series of real terms.

Indeterminate series are unsafe; and, by reason of their slow convergence, they are worthless.

§ 6. IMAGINARY SERIES.

A SERIES whose terms are part or all imaginary is an *imaginary series*. If each term of the series T_1, T_2, ... be resolved into its two components P_1, $Q_1 i$; P_2, $Q_2 i$; ..., the two series s', $\equiv P_1 + P_2 + \cdots$, and $s''i$, $\equiv Q_1 i + Q_2 i + \cdots$, are the components of s, and $s = s' + s''i$.

The moduli of the several terms taken in order form the *series of moduli*, a series of real positive numbers, $\equiv \overset{+}{\sqrt{}}(P_1^2 + Q_1^2) \cdots$.

THEOR. 14. *If for any imaginary series the series of moduli be convergent, the imaginary series is convergent.*

For $\because$ the series s', s'' have their terms when made positive not greater than the corresponding terms of the convergent series of moduli, [$^+P \ngtr \overset{+}{\sqrt{}}(P^2 + Q^2), \cdots$

$\therefore$ s', s'' are convergent, [ths. 10, 13

$\therefore$ s, $= s' + s''i$, is convergent. Q.E.D.

THEOR. 15. *If for any imaginary series the series of moduli be divergent, the imaginary series is non-convergent.*

For $\because$ $^+s' + {}^+s''$, the sum of the component series s', $s''i$, with all their terms made real and positive, is not less than the divergent series of moduli, [$^+P + {}^+Q \nless \overset{+}{\sqrt{}}(P^2 + Q^2)$

$\therefore$ one or both of the series $^+s'$, $^+s''$ are divergent,

$\therefore$ one or both of the series s', s'' are non-convergent, [th. 12 cr. 1

$\therefore$ s, $= s' + s''i$, is non-convergent. Q.E.D.

COR. *If* s *be* $\left\{ \begin{matrix} \textit{convergent}, \\ \textit{non-convergent}, \end{matrix} \right.$ *so is its series of moduli.*

NOTE. Theors. 14, 15, when applied to series of real numbers, become theor. 13 and its converse, since the modulus of a real number is that number taken positive.

In the light of theors. 14, 15, it appears that theors. 7–11, with their notes and corollaries, apply to series with imaginary terms, and that those theorems are general for all series. Theor. 16 shows that every series to rising powers of a variable has a radius of convergence.

THEOR. 16 (ABEL'S THEOREM). *If a series,* $A_0+A_1z+A_2z^2+\cdots$, *arranged to rising powers of a variable* z, *be* $\left\{\begin{array}{l}\textit{convergent}\\ \textit{non-convergent}\end{array}\right.$ *when* mod z $=$ *a constant* r, *it is* $\left\{\begin{array}{l}\textit{convergent}\\ \textit{non-converg't}\end{array}\right.$, *whenever mod* z $\left\{\begin{array}{l}\not> r.\\ \not< r.\end{array}\right.$

For $\because$ $A_0+A_1\cdot z+A_2\cdot z^2+\cdots$ is $\left\{\begin{array}{l}\text{convergent}\\ \text{non-convergent}\end{array}\right.$ when mod $z=r$, [hyp.

$\therefore$ mod A_0 + mod $A_1\cdot$ mod z + mod $A_2\cdot\overline{\text{mod } z}^2+\cdots$ is $\left\{\begin{array}{l}\text{convergent}\\ \text{divergent}\end{array}\right.$ when mod $z=r$; [th. 15 cr.

$\therefore$ mod A_0 + mod $A_1\cdot$ mod z + mod $A_2\cdot\overline{\text{mod } z}^2+\cdots$ is $\left\{\begin{array}{l}\text{convergent}\\ \text{divergent}\end{array}\right.$ whenever mod $z\left\{\begin{array}{l}\not> r.\\ \not< r.\end{array}\right.$

$\therefore$ s, $=A_0+A_1z+A_2z^2+\cdots$, is $\left\{\begin{array}{l}\text{convergent}\\ \text{non-convergent}\end{array}\right.$ whenever mod $z\left\{\begin{array}{l}\not> r.\\ \not< r.\end{array}\right.$ Q.E.D. [ths. 14, 15

COR. *If, in a series arranged to rising powers of* z, *mod* z *increase from* 0 *to* ∞, *the series is* $\left\{\begin{array}{l}\textit{convergent}\\ \textit{in doubt}\\ \textit{divergent}\end{array}\right.$ *when mod* z $\left\{\begin{array}{l}<\\ =r.\\ >\end{array}\right.$

[r a constant, called the *radius of convergence* of the series. In most series r is lim ratio mod A_n : mod A_{n+1}. [th. 11 nt. 5

NOTE. GRAPHIC REPRESENTATION: Denote by z the representative point of any number z; *i.e.*, the extremity of that vector from the origin whose ratio to the unit-line is z; and so for other numbers. Let A_0, A_1z, A_2z^2, ... be any series arranged to rising powers of z; and from O as centre, with radius equal to the radius of convergence of the series, draw a circle; this circle, called the *circle of convergence*, embraces the region $\left\{\begin{array}{l}\text{within}\\ \text{upon}\\ \text{without}\end{array}\right.$ which z lies when the series is $\left\{\begin{array}{l}\text{convergent.}\\ \text{in doubt.}\\ \text{divergent.}\end{array}\right.$

If a series be arranged to rising powers of $(z-a)$, then the circle of convergence has a for centre and r for radius, and the series is $\left\{\begin{array}{l}\text{convergent}\\ \text{in doubt}\\ \text{divergent}\end{array}\right.$ when z lies $\left\{\begin{array}{l}\text{within}\\ \text{upon}\\ \text{without}\end{array}\right.$ this circle;

for mod $(z-a)\left\{\begin{array}{l}<r\\ =r.\\ >r.\end{array}\right.$

THEOR. 17. *In a series arranged to rising powers of a variable* z, *if* mod z *be less than the radius of convergence of the series, an increment can be given to* z *so small that the increment of the series shall be less than any assigned number.*

For let $s \equiv A_0 + A_1 z + A_2 z^2 + \cdots$, take mod z less than r, the radius of convergence, and to z give an increment h so small that mod $(z+h) < r$;

then $\because$ s and s + inc s, $\equiv A_0 + A_1(z+h) + A_2(z+h)^2 + \cdots$, are both convergent series, [hyp.

$\therefore$ inc s, $\equiv h\left(A_1 + A_2\dfrac{(z+h)^2 - z^2}{h} + \cdots\right)$, is convergent;

$\therefore$ inc s : h is a convergent series when h is finite; [th. 8

and $\because$ h may approach 0 so that $[(z+h)^n - z^n] : h$ is larger than but approaches nz^{n-1}, [bin. th.

$\therefore$ inc s : $h \doteq$ a finite limit when $h \doteq 0$; [th. 10

$\therefore$ inc s, $= h \cdot$ a finite number, $\doteq 0$ when $h \doteq 0$. Q.E.D.

COR. 1. $D_z s, = A_1 + 2A_2 z + 3A_3 z^2 + \cdots, D_z^2 s, D_z^3 s, \cdots$, *are all series whose common radius of convergence is* r.

COR. 2. *For all values of mod* z $\left\{\begin{matrix}\textit{smaller}\\ \textit{larger}\end{matrix}\right.$ *than* r, *the series-function* $A_0 + A_1 z + A_2 z^2 + \cdots$ *is* $\left\{\begin{matrix}\textit{a finite continuous one-value}\\ \textit{an infinite or indeterminate}\end{matrix}\right.$ *function of* z.

If s be a series to rising powers of a variable z, and z be a finite function of z that is equal to s for continuous values of z from 0 to r, but unequal for a value of z larger than r, then s and z are discontinuous when $z \equiv r$ [theory of functions], and r is the radius of convergence of s and the smallest value of z for which z is discontinuous.

In the graphic representation of imaginaries, if the *points of discontinuity*, $a, b, c, \cdots$ of the functions $(a-z)^p(b-z)^q(c-z)^r \cdots$ ($p, q, r, \cdots$ any fractions or negative integers) be platted, and the function be equal to a series to rising powers of $z-k$, then, with k as the centre of convergence, the radius of convergence of the series is the distance from k to the nearest point of discontinuity.

§ 7. EXPANSION OF FUNCTIONS IN INFINITE SERIES.

If $\text{z} = \text{T}_1 + \text{T}_2 + \text{T}_3 + \cdots$ [z, T_1, T_2, T_3 ⋯ functions of z] for all values of z that make the second member a convergent series, the series is an *expansion* of z in functions of z.

An ordinary function of a variable cannot, in general, be equal to any one infinite series for all values of that variable.

E.g., if z be a variable that increases from 0 to ∞, then the

$$\text{fraction } \frac{1}{1-z} \text{ is} \begin{cases} \text{finite and positive} \\ \text{infinite} \\ \text{finite and negative} \end{cases} \text{when } z \begin{cases} <1; \\ =1; \\ >1; \end{cases}$$

but $\because$ the series is infinite when $z > 1$,

$\therefore$ the series $1+z+z^2+\cdots$, which equals the fraction for all values of z from 0 to 1, ceases to equal it when $z > 1$.

So, the series $-z-z^2-z^3\cdots$, wherein $z=1:x$, is an expansion of $(1-x)^{-1}, \equiv -z(1-z)^{-1}$;

and the two are equal when $z<1$,

i.e, when $x>1$;

but the series is divergent, and the two are unequal when $z>1$.

$$\text{So, the radical } \sqrt{(1-z)} \text{ is} \begin{cases} \text{real} \\ \text{zero} \\ \text{imaginary} \end{cases} \text{when } z \begin{cases} <1; \\ =1; \\ >1; \end{cases}$$

and it is shown later that an expansion of $\sqrt{(1-z)}$ is

$1-\frac{1}{2}z-\frac{1}{8}z^2-\frac{1}{16}z^3-\cdots$; [bin. th.

but this equality is impossible when $z>1$;

for the series-function remains real for all real values of z, and the radical becomes imaginary when $z>1$.

So, if n be any $\left\{\begin{matrix}\text{negative integer,} \\ \text{fraction,}\end{matrix}\right.$ the $\left\{\begin{matrix}\text{fraction} \\ \text{radical}\end{matrix}\right.$ $(a-z)^n$ may be expanded into the series

$$a^n - na^{n-1}z + \frac{n(n-1)}{2!}a^{n-2}z^2 - \cdots, \qquad \text{[bin. th.}$$

whose radius of convergence is a; [th. 11, nt. 5

then the $\left\{\begin{matrix}\text{fraction} \\ \text{radical}\end{matrix}\right.$ is not equal to the series when $z \geqslant a$.

So, if p, q, r be any fractions or negative integers, and if $\text{z} = (a-z)^p(b-z)^q(c-z)^r\cdots$, then z cannot equal a series to rising powers of z when z is larger than the smallest of the numbers a, b, c, ⋯.

THEOR. 18. *The sum of an infinite series* $A_0 + A_1 X + A_2 X^2 + \cdots$, *whose radius of convergence is greater than* 0, *approaches the limit* A_0 *when* $X \doteq 0$.

For $\because$ $A_1 + A_2 x + A_3 x^2 + \cdots$ has the same radius of convergence as the given series,

$\therefore$ it is convergent for small values of x,

$\therefore$ the product $x(A_1 + A_2 x + A_3 x^2 + \cdots) \doteq 0$, when $x \doteq 0$,

$\therefore$ $A_0 + A_1 x + A_2 x^2 + \cdots \doteq A_0$ when $x \doteq 0$. Q.E.D.

COR. *In the infinite series* $A_0 + A_1 X + A_2 X^2 + \cdots + A_r X^r + \cdots$, X *may be made so* $\left\{ \begin{matrix} small \\ large \end{matrix} \right.$ *that* $A_r X^r$ *shall be any number of times larger than the sum of all the terms of* $\left\{ \begin{matrix} higher \\ lower \end{matrix} \right.$ *degree.*

THEOR. 19. *If two series, arranged to rising powers of any same variable, be equal for all values of the variable that make them both convergent, the coefficients of like powers of the variable are equal.*

Let $A_0 + A_1 x + A_2 x^2 + \cdots = A_0' + A_1' x + A_2' x^2 + \cdots$, when $x < r$, wherein if the series have different radii of convergence, r is the least of the two;

then will $A_0 = A_0'$, $A_1 = A_1'$, $A_2 = A_2'$, $\cdots$.

For $\because$ the two series are equal when $x < r$, [hyp.

$\therefore$ they approach equal limits when $x \doteq 0$;

i.e., $A_0 = A_0'$. [th.18

$\therefore$ $A_1 x + A_2 x^2 + \cdots = A_1' x + A_2' x^2 + \cdots$ when $x < r$.

$\therefore$ $A_1 + A_2 x + \cdots = A_1' + A_2' x + \cdots$ when $x < r$. [div. by x

$\therefore$ $A_1 = A_1'$. [as above

So $A_2 = A_2'$, and so on. Q.E.D.

COR. *No function* X *has more than one expansion to ascending powers of a given variable* x.

For if possible let there be two separate expansions;

then $\because$ each expansion is equal to X when it is convergent, [df.

$\therefore$ the two expansions are equal to each other when both are convergent,

$\therefore$ their coefficients are equal, and the two are identical. Q.E.D. [th.

§ 8. METHOD OF UNKNOWN COEFFICIENTS.

The *method of unknown coefficients* is used for the purpose of changing a function from one form to another. It consists in equating the given function to a function of the required form with unknown coefficients, and then finding such values of these coefficients, if possible, as shall make the two members identical.

EXPANSION OF FRACTIONS.

PROB. 4. TO EXPAND A FRACTION INTO A SERIES.

Put the fraction equal to a series arranged to the rising powers of some letter in the denominator of the fraction, and with unknown coefficients.

Free the equation from fractions.

Equate the coefficients of the like powers of the letter of arrangement in the two members, each to each, and solve the equations thus found for the unknown coefficients. [th. 19

E.g., put $\dfrac{1+2x}{1-3x+5x^2} \equiv A + Bx + Cx^2 + Dx^3 + \cdots;$

then $\because\ 1+2x = A + \left.\begin{array}{r} B \\ -3A \end{array}\right| x + \left.\begin{array}{r} C \\ -3B \\ +5A \end{array}\right| x^2 + \left.\begin{array}{r} D \\ -3C \\ +5B \end{array}\right| x^3 + \cdots,$

$$A = 1,\quad B - 3A = 2,\quad C - 3B + 5A = 0,\ \cdots;$$

$$\therefore\ A = 1,\quad B = 2 + 3A = 5,\quad C = 3B - 5A = 10,\ \cdots;$$

and the series is $1 + 5x + 10x^2 + 5x^3 - 35x^4 \cdots,$

wherein every coefficient after the second equals three times the coefficient next before less five times the one before that.

RECURRING SERIES.

A series like that in the example above is a *recurring series;* it is a compound geometric progression, each of whose terms is the sum of the products of the two or more next preceding terms by constant multipliers. The group of multipliers is the *scale*.

E.g., $C = 3B - 5A,\ D = 3C - 5B,\ E = 3D - 5C, \cdots,$ [ex. pr. 4

and $(3, -5)$ is the scale for the series of coefficients;

and of the series $1,\ 5x,\ 10x^2,\ 5x^3,\ {}^{-}35x^4, \cdots$, the scale is $3x,\ {}^{-}5x^2$, and the sum is the fraction $\dfrac{1+2x}{1-3x+5x^2}$.

PROB. 5. TO FIND THE SCALE AND SUM OF A RECURRING SERIES.

(a) *Scale of two terms*, m, n.

Write $T_3 = mT_2 + nT_1$, $T_4 = mT_3 + nT_2$, $T_5 = mT_4 + nT_3$.

Solve the first two equations for m, n, *and test the values thus found by the third equation.*

Write $s = \dfrac{T_1(1-m)+T_2}{1-m-n}$.

For $\because$ $s = T_1 + T_2 + T_3 + T_4 + \cdots$

$= T_1 + T_2 + (mT_2 + nT_1) + (mT_3 + nT_2) + \cdots$

$= T_1 + T_2 + m(T_2 + T_3 + T_4 + \cdots) + n(T_1 + T_2 + T_3 + \cdots)$

$= T_1 + T_2 + m(s - T_1) + ns$,

$\therefore$ $s = \dfrac{T_1(1-m)+T_2}{1-m-n}$. Q.E.D. [sol. for s

E.g., to find the scale and sum of the recurring series

$1 + 5x + 10x^2 + 5x^3 - 35x^4 \cdots$.

Write $10x^2 = m \cdot 5x + n \cdot 1$ and $5x^3 = m \cdot 10x^2 + n \cdot 5x$; solve for m, n; and test by equation $-35x^4 = m \cdot 5x^3 + n \cdot 10x^2$;

then $m = 3x$, $n = -5x^2$;

and $s = \dfrac{1 - 3x + 5x}{1 - 3x + 5x^2} = \dfrac{1 + 2x}{1 - 3x + 5x^2}$.

(b) *Scale of three terms*, m, n, p.

Write $T_4 = mT_3 + nT_2 + pT_1$, $T_5 = mT_4 + nT_3 + pT_2$,

$T_6 = mT_5 + nT_4 + pT_3$, $T_7 = mT_6 + nT_5 + pT_4$.

Solve the first three equations for m, n, p, *and test by the fourth.*

Write $s = \dfrac{T_1(1-m-n) + T_2(1-m) + T_3}{1-m-n-p}$.

For $\because$ $s = T_1 + T_2 + T_3 + T_4 + \cdots$

$= T_1 + T_2 + T_3 + (mT_3 + nT_2 + pT_1)$

$+ (mT_4 + nT_3 + pT_2) + \cdots$

$= T_1 + T_2 + T_3 + m(T_3 + T_4 + T_5 + \cdots)$

$+ n(T_2 + T_3 + T_4 + \cdots) + p(T_1 + T_2 + T_3 + \cdots)$

$= T_1 + T_2 + T_3 + m(s - T_1 - T_2) + n(s - T_1) + ps$,

$\therefore$ $s = \dfrac{T_1(1-m-n) + T_2(1-m) + T_3}{1-m-n-p}$. Q.E.D.

(c) *So, for scale of four or more terms.*

EXPANSION OF SURDS.

PROB. 6. TO EXPAND A SURD INTO A SERIES:

Put the surd equal to a series arranged to the rising powers of some letter in the surd, and with unknown coefficients.

Free the equation from radicals.

Equate the coefficients of the like powers of the letter of arrangement, each to each, and solve the equations thus found for the unknown coefficients. [th.19

E.g., to expand $\sqrt{(a^2+bx)}$:

Put $\sqrt{(a^2+bx)} \equiv A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \cdots$;

then $\because$ $a^2+bx = A^2 + 2AB\,x + \left.\begin{matrix}2AC\\B^2\end{matrix}\right| x^2 + \left.\begin{matrix}2AD\\2BC\end{matrix}\right| x^3 + \left.\begin{matrix}2AE\\2BD\\C^2\end{matrix}\right| x^4 + \cdots,$

$$\therefore\ A^2 = a^2,\quad 2AB = b,\quad 2AC + B^2 = 0,\quad 2AD + 2BC = 0,\ \cdots;$$

$$\therefore\ A = a,\quad B = \frac{b}{2a},\quad C = \frac{-b^2}{8a^3},\quad D = \frac{b^3}{16a^5},\ \cdots,$$

and $$\sqrt{(a^2+bx)} = a + \frac{bx}{2a} - \frac{b^2x^2}{8a^3} + \frac{b^3x^3}{16a^5} - \frac{5b^4x^4}{128a^7} + \cdots.$$

So, $\sqrt{5} = \sqrt{(4+1)} = 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} - \frac{5}{16384} + \cdots.$

So, put $\sqrt[3]{(a^3+bx)} = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$;

then $\because$ $a^3+bx = A^3 + 3A^2B \left| x + \begin{matrix}3AB^2\\+3A^2C\end{matrix} \right| x^2 + \begin{matrix}B^3\\+3A^2D\\+6ABC\end{matrix} \left| x^3 + \begin{matrix}3A^2E\\+3AC^2\\+3B^2C\\+6ABD\end{matrix} \right| x^4 + \cdots,$

$$\therefore\ A = a,\quad B = \frac{b}{3a^2},\quad C = -\frac{b^2}{9a^5},\quad D = \frac{5b^3}{81a^8},\quad E = -\frac{10b^4}{243a^{11}}.$$

$$\therefore\ \sqrt[3]{(a^3+bx)} = a + \frac{bx}{3a^2} - \frac{b^2x^2}{9a^5} + \frac{5b^3x^3}{81a^8} - \frac{10b^4x^4}{243a^{11}} + \cdots.$$

So, $\sqrt[3]{9} = \sqrt[3]{(8+1)} = 2 + \frac{1}{12} - \frac{1}{288} + \frac{5}{20736} - \cdots.$

NOTE. This method of expanding $(1+x)^{\frac{p}{q}}$ shows that a series $A + Bx + Cx^2 + \cdots$ exists whose qth power is identically $(1+x)^p$; and so this series, when convergent, is a qth root of $(1+x)^p$. There are q such series corresponding to the q, qth roots of unity. [X. th. ??

RESOLUTION OF FRACTIONS.

PROB. 7. TO RESOLVE INTO A SUM OF PARTIAL FRACTIONS A FRACTION WHOSE TERMS ARE ENTIRE FUNCTIONS OF ANY ELEMENT:

If the degree of the numerator be not lower than that of the denominator, reduce the fraction to a mixed number.

Resolve the denominator of the fraction into its prime factors.

Equate the fraction to a set of fractions formed as follows:

For every prime factor not repeated write a fraction whose denominator is that prime factor; and for any prime factor repeated k *times write* k *fractions whose denominators are the first, second, third, ⋯* k*th powers of the factor.*

For the numerator of any fraction write an entire function of the given element with unknown coefficients, and of degree lower by unity than the prime factor that enters into its denominator.

Free this equation from fractions.

Equate the coefficients of the several powers of the letter of arrangement, each to each, and solve the equations thus found for the unknown coefficients of the numerators.

E.g., to resolve $\dfrac{1}{x^3-1}$: [$x^3-1=(x-1)(x^2+x+1)$

Write $\dfrac{1}{x^3-1}\equiv\dfrac{\mathrm{A}x+\mathrm{B}}{x^2+x+1}+\dfrac{\mathrm{C}}{x-1}$;

then $\because\ 1=(\mathrm{A}+\mathrm{C})x^2+(-\mathrm{A}+\mathrm{B}+\mathrm{C})x-\mathrm{B}+\mathrm{C}$ [free fr. frac.

$\therefore\ \mathrm{A}+\mathrm{C}=0,\quad -\mathrm{A}+\mathrm{B}+\mathrm{C}=0,\quad -\mathrm{B}+\mathrm{C}=1$

$\therefore\ \mathrm{A}=-\frac{1}{3},\quad \mathrm{B}=-\frac{2}{3},\quad \mathrm{C}=\frac{1}{3},$

and $$\frac{1}{x^3-1}=-\frac{x+2}{3(x^2+x+1)}+\frac{1}{3(x-1)}.$$

So, write $\dfrac{2x^2-10x+14}{(x-1)(x-2)(x-3)}\equiv\dfrac{\mathrm{A}}{x-1}+\dfrac{\mathrm{B}}{x-2}+\dfrac{\mathrm{C}}{x-3}$;

then $\because\ 2x^2-10x+14$

$$=\mathrm{A}(x^2-5x+6)+\mathrm{B}(x^2-4x+3)+\mathrm{C}(x^2-3x+2),$$

$\therefore\ \mathrm{A}+\mathrm{B}+\mathrm{C}=2,\quad 5\mathrm{A}+4\mathrm{B}+3\mathrm{C}=10,\quad 6\mathrm{A}+3\mathrm{B}+2\mathrm{C}=14,$

$\therefore\ \mathrm{A}=3,\quad \mathrm{B}={}^-2,\quad \mathrm{C}=1,$

and $$\frac{2x^2-10x+14}{(x-1)(x-2)(x-3)}=\frac{3}{x-1}-\frac{2}{x-2}+\frac{1}{x-3}.$$

Since, as appears from Note 1 below, the identity

$$2x^2 - 10x + 14$$
$$\equiv A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

holds true for every value of x, it is more readily solved as follows :

Put $x = 1$; then $2 - 10 + 14 = A \cdot {}^-1 \cdot {}^-2$, and $A = 3$.

Put $x = 2$; then $B = {}^-2$.

Put $x = 3$; then $C = 1$.

So, write $\dfrac{2x^2 + x + 3}{(x+1)^3} \equiv \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$;

then $A = 2,\ B = {}^-3,\ C = 4,$

and $$\frac{2x^2 + x + 3}{(x+1)^3} = \frac{2}{x+1} - \frac{3}{(x+1)^2} + \frac{4}{(x+1)^3}.$$

This fraction may also be resolved as follows :

$\because \dfrac{2x^2 + x + 3}{(x+1)^3} \equiv \dfrac{2x-1}{(x+1)^2} + \dfrac{4}{(x+1)^3}$, [div. by $x+1$

and $\because \dfrac{2x-1}{(x+1)^2} \equiv \dfrac{2}{x+1} - \dfrac{3}{(x+1)^2}$, [div. by $x+1$

$\therefore \dfrac{2x^2 + x + 3}{(x+1)^3} \equiv \dfrac{2}{x+1} - \dfrac{3}{(x+1)^2} + \dfrac{4}{(x+1)^3}$, as before.

So, write $\dfrac{4x^4 + 3x^3 - x^2 - 4x - 1}{(x+1)(x^2+x+1)^2} \equiv \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2+x+1} + \dfrac{Dx + E}{(x^2+x+1)^2}$;

then $4x^4 + 3x^3 - x^2 - 4x - 1$
$$= A(x^2+x+1)^2 + (\overline{Bx+C} \cdot \overline{x^2+x+1} + Dx + E)(x+1).$$

Put $x = -1$; then $A = 3$.

And $\because (Bx + C)(x^2 + x + 1) + Dx + E$ [repl. A, div. by $x+1$
$$= [4x^4 + 3x^3 - x^2 - 4x - 1 - 3(x^2+x+1)^2] : (x+1)$$
$$= x^3 - 4x^2 - 6x - 4,$$

$\therefore Bx + C = x - 5,\ Dx + E = -2x + 1,$ [div. by x^2+x+1

and $$\frac{4x^4 + 3x^3 - x^2 - 4x - 1}{(x+1)(x^2+x+1)^2} = \frac{3}{x+1} + \frac{x-5}{x^2+x+1} - \frac{2x-1}{(x^2+x+1)^2}.$$

The division without remainder by $(x+1)$ is a useful check.

NOTE 1. When unknown coefficients are got by giving special values to a variable x, the work does not of itself show whether any development of the proposed form be possible, but only shows what the coefficients must be if the development be possible. That every fraction is resolvable into partial fractions as here proposed appears, however, as follows.

Let the given fraction be $\frac{\text{U}}{\text{VW}}$, wherein U, V, W are entire and prime to one another; let x_1 be any value of x, for which $\text{V}=0$; let U_1, W_1 be the constants that U, W become when the variable x is replaced by the particular value x_1;

then $\quad \frac{\text{U}}{\text{VW}}, \equiv \frac{\text{U}_1}{\text{W}_1 \cdot \text{V}} + \frac{\text{W}_1 \cdot \text{U} - \text{U}_1 \cdot \text{W}}{\text{W}_1 \cdot \text{WV}}, = \frac{\text{A}}{\text{V}} + \frac{\text{U}'}{\text{V}'\text{W}},$

wherein $\text{A} \equiv \frac{\text{U}_1}{\text{W}_1}$, $\text{U}' \equiv \frac{\text{W}_1 \cdot \text{U} - \text{U}_1 \cdot \text{W}}{\text{W}_1} : (x-x_1)$, $\text{V}' \equiv \text{V} : (x-x_1)$;

for $\because$ the entire expressions $\text{W}_1\text{U} - \text{U}_1\text{W}$, V, $=0$ when $x=x_1$,
$\therefore$ each of them is divisible by $x - x_1$. [XI. th. 4.

So, $\quad \frac{\text{U}'}{\text{V}'\text{W}} = \frac{\text{B}}{\text{V}'} + \frac{\text{U}''}{\text{V}''\text{W}}, \Big[\text{B} \equiv \frac{\text{U}_2'}{\text{W}}, \ \text{U}'' \equiv \frac{\text{W}_2 \cdot \text{U}' - \text{U}_2' \cdot \text{W}}{\text{W}} : (x-x_2)$

wherein x_2 is any value of x_1 for which $\text{V}'=0$, $\text{U}'=\text{U}_2'$, $\text{W}=\text{W}_2$; ...

$$\therefore \frac{\text{U}}{\text{VW}} = \left(\frac{\text{A}}{\text{V}} + \frac{\text{B}}{\text{V}'} + \frac{\text{C}}{\text{V}''} + \cdots\right) + \frac{\text{R}}{\text{W}}$$

$$= \frac{\text{Q}}{\text{V}} + \frac{\text{R}}{\text{W}}; \ \Big[\text{Q} \equiv \text{A} + \text{B} \cdot \overline{x-x_1} + \text{C} \cdot \overline{x-x_1} \cdot \overline{x-x_2} + \cdots$$

and the given fraction is resolved as proposed.

If the denominator VW have three or more factors, then one of them, say V, can be factored again, and so on.

If V be a power v^m, then $\frac{\text{Q}}{\text{V}}$ is resolvable by division into $\frac{q}{v^m} + \frac{r}{v^{m-1}} + \cdots + \frac{t}{r}$.

E.g., above, where $\text{V} = (x+1)^3$.

NOTE 2. One of the uses of Prob. 7 is in the integration of rational fractions:

$$E.g., \int \frac{2x^2 - 10x + 14}{(x-1)(x-2)(x-3)} dx = \int \left(\frac{3}{x-1} - \frac{2}{x-2} + \frac{1}{x-3}\right) dx$$

$$= 3 \log (x-1) - 2 \log (x-2) + \log (x-3) + \text{a constant.}$$

REVERSION OF SERIES.

PROB. 8. IF A VARIABLE BE EQUAL TO A SERIES OF POWERS OF ANOTHER VARIABLE, TO FIND THE VALUE OF THE SECOND VARIABLE IN TERMS OF THE FIRST:

Put the letter of arrangement of the given series equal to a new series arranged to powers of the required letter of arrangement with unknown coefficients, and in the new series replace the new letter of arrangement by the given series.

Equate the coefficients of the like powers of the old letter of arrangement, each to each, and solve the equations thus found for the unknown coefficients.

E.g., to revert the series $y = ax + bx^2 + cx^3 + \cdots$:

Put $x \equiv \text{A}y + \text{B}y^2 + \text{C}y^3 + \cdots$, and replace $y, y^2, y^3, \cdots$ by $(ax + bx^2 + cx^3 + \cdots)$, $(ax + bx^2 + cx^3 + \cdots)^2, \cdots$;

then $\because$ $$x = \text{A}ax + \left.\begin{matrix}\text{A}b \\ +\text{B}a^2\end{matrix}\right| x^2 + \left.\begin{matrix}\text{AC} \\ +2\text{B}ab \\ +\text{C}a^3\end{matrix}\right| x^3 + \cdots,$$

$$\therefore\ \text{A}a = 1,\quad \text{A}b + \text{B}a^2 = 0,\quad \text{AC} + 2\text{B}ab + \text{C}a^3 = 0, \cdots.$$

$$\therefore\ \text{A} = \frac{1}{a},\quad \text{B} = -\frac{b}{a^3},\quad \text{C} = \frac{2b^2 - ac}{a^5},\quad \cdots,$$

and $$x = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 + \cdots.$$

So, to revert the series $y = m + ax + bx^2 + cx^3 + \cdots$:

then $\because$ $y - m = ax + bx^2 + cx^3 + \cdots$,

$$\therefore\ x = \frac{1}{a}(y - m) - \frac{b}{a^3}(y - m)^2 + \frac{2b^2 - ac}{a^5}(y - m)^3 + \cdots.$$

So, to revert the series $y = ax^2 + bx^4 + cx^6 + \cdots$:

then $$x^2 = \frac{1}{a}y - \frac{b}{a^3}y^2 + \frac{2b^2 - ac}{a^5}y^3 + \cdots.$$

So, to revert the series $y = ax + bx^3 + cx^5 + \cdots$:

Put $x \equiv \text{A}y + \text{B}y^3 + \text{C}y^5 + \cdots$;

then $\because$ $$x = \text{A}ax + \left.\begin{matrix}\text{A}b \\ +\text{B}a^3\end{matrix}\right| x^3 + \left.\begin{matrix}\text{AC} \\ +3\text{B}a^2b \\ +\text{C}a^5\end{matrix}\right| x^5 + \cdots,$$

$$\therefore\ \text{A}a = 1,\quad \text{A}b + \text{B}a^3 = 0,\quad \text{AC} + 3\text{B}a^2b + \text{C}a^5 = 0, \cdots;$$

and $$x = \frac{1}{a}y - \frac{b}{a^4}y^3 + \frac{3b^2 - ac}{a^7}y^5 + \cdots.$$

§ 9. BINOMIAL THEOREM.

THEOR. 20. *If* a + b *be any binomial, and* n *any real number, then*

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$
$$+\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}a^{n-r}b^r + \cdots. \quad [\text{V. th.1}$$

For, put $x \equiv \overline{b:a}$;

then $(a+b)^n = a^n(1+\overline{b:a})^n = a^n(1+x)^n$.

(*a*) n *commensurable.*

Put $(1+x)^n \equiv 1 + Bx + Cx^2 + Dx^3 + \cdots$; [B, C, D, ⋯ unkn., pr. 6 nt.

then $n(1+x)^{n-1} = B + 2Cx + 3Dx^2 + \cdots$, [VII. th. 17, cr. 1;

and $n(1+x)^n = B + \left.\begin{matrix}2C\\+B\end{matrix}\right| x + \left.\begin{matrix}3D\\+2C\end{matrix}\right| x^2 + \cdots$. [mult. by $1+x$

But $\because n(1+x)^n = n + nBx + nCx^2 + nDx^3 + \cdots$, [above

$\therefore B + \left.\begin{matrix}2C\\+B\end{matrix}\right| x + \left.\begin{matrix}3D\\+2C\end{matrix}\right| x^2 + \cdots = n + nBx + nCx^2 + nDx^3 + \cdots$,

$\therefore B = n,\ 2C + B = nB,\ 3D + 2C = nC,\ \cdots$,

and $B = n,\ C = \dfrac{n(n-1)}{2!},\ D = \dfrac{n(n-1)(n-2)}{3!},\ \cdots$.

$\therefore (1+x)^n$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots,$$

and $(a+b)^n = a^n + na^{n-1}b + \dfrac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$. Q.E.D.

(*b*) n *incommensurable*, a case of limits.

NOTE. Although the form of the series does not depend on the ratio $b:a$, yet the series is worthless unless convergent.

E.g., $\sqrt{5} = \sqrt{(4+1)} = 4^{\frac{1}{2}} + \frac{1}{2}\cdot 4^{-\frac{1}{2}} - \frac{1}{8}\cdot 4^{-\frac{3}{2}} + \frac{1}{16}\cdot 4^{-\frac{5}{2}} - \cdots$,

and the convergents are 2, $2\frac{1}{4}$, $2\frac{15}{64}$, $2\frac{121}{512}$, ⋯;

but $\sqrt{5} = \sqrt{(1+4)} = 1 + \frac{1}{2}\cdot 4 - \frac{1}{8}\cdot 4^2 + \frac{1}{16}\cdot 4^3 -, \cdots$,

and the convergents are 1, 3, 1, 5, ⋯, which are useless.

So, $\sqrt{3} = \sqrt{(4-1)} \doteq 2,\ 1\frac{3}{4},\ 1\frac{47}{64},\ 1\frac{375}{512},\ \cdots$;

but $\sqrt{-3} = \sqrt{(1-4)} \doteq 1,\ ^-1,\ ^-3,\ ^-7,\ \cdots$, which is absurd.

THEOR. 21. *The series* $1 + n + \frac{n(n-1)}{2!} + \cdots$

$$+ \frac{n(n-1)\cdots(n-r+1)}{r!} + \frac{n(n-1)\cdots(n-r+1)(n-r)}{(r+1)!} + \cdots$$

is convergent if n *be positive.*

1. r *may be made so large that* $T_{r+2} \cdot (r+1)^{n+1} \lessdot T_{r+1} \cdot r^{n+1}$.

For $$[T_{r+2} \cdot (r+1)^{n+1}] : (T_{r+1} \cdot r^{n+1}) = \left(\frac{r+1}{r}\right)^{n+1} \cdot \left(\frac{n-r}{r+1}\right)$$

$$= -\left(1 + \frac{1}{r}\right)^n \cdot \left(1 - \frac{n}{r}\right)$$

$$= -\left(1 + \frac{n}{r} + \frac{n(n-1)}{2r^2} + \cdots\right) \cdot \left(1 - \frac{n}{r}\right), \qquad \text{[bin. th.}$$

$$= -\left(1 - \frac{n(n+1)}{2}\,\frac{1}{r^2} + \cdots\right) = h, \text{ say;}$$

and $\frac{1}{r}$ may be taken so small that $h \lessdot 1$. Q.E.D. [th. 18, cr.

2. *The series is convergent.*

For ∵ after r becomes larger than some fixed finite value r', each product $T_{r+1} \cdot r^{n+1}$ is smaller than the product before it, [1

and ∵ $T_{r'+1} \cdot r'^{n+1}$ is some finite number, say k,

∴ $T_{r+1} \lessdot k : r^{n+1}$,

∴ the series is convergent. Q.E.D. [th. 11, nt. 3

COR. *The expansion of* $(a + a)^n$ *is* $\left\{\begin{matrix}\textit{convergent}\\ \textit{non-convergent}\end{matrix}\right.$ *if* n *be* $\left\{\begin{matrix}\textit{positive;}\\ \textit{negative;}\end{matrix}\right.$ *and that of* $(a \pm b)^n$ *is* $\left\{\begin{matrix}\textit{convergent}\\ \textit{non-convergent}\end{matrix}\right.$ *if* $a \left\{\begin{matrix} > \\ < \end{matrix}\right. b$.

NOTE. The expansion of $(a + a)^n$ is indeterminate if n lie between 0 and 1; for then the successive terms of the series do not grow larger, and are alternately positive and negative.

But if the negative terms be made positive, the series is the expansion of $(a - a)^n$, whose value, a negative power of 0, is infinite.

The expansion of $(a + a)^n$ is divergent if $n < -1$; for then the successive terms of the series grow larger and larger.

PROB. 9. TO EXPAND A POWER OF A BINOMIAL:

Reduce the given expression to the type-form $(\mathfrak{a}+\mathfrak{b})^{\mathfrak{n}}$ *and apply the binomial formula.* [th. 20

E.g., $$(x-y)^4=x^4+4x^3(-y)+\frac{4\cdot 3}{2\,!}x^2(-y)^2+\frac{4\cdot 3\cdot 2}{3\,!}x(-y)^3+(-y)^4,$$
$$=x^4-4x^3y+6x^2y^2-4xy^3+y^4.$$

So, $$(2a-3b)^{-4}=(2a)^{-4}+{}^-4\cdot(2a)^{-5}\cdot(-3b)+\frac{{}^-4\cdot{}^-5}{2\,!}\cdot(2a)^{-6}\cdot(-3b)^2$$
$$+\cdots+\frac{{}^-4\cdot{}^-5\cdot{}^-6\cdots{}^-(r+3)}{r\,!}\cdot 2a^{-(r+4)}\cdot(-3b)^r+\cdots,$$
$$=\frac{1}{(2a)^4}+\frac{4\cdot(3b)}{(2a)^5}+\frac{4\cdot 5\cdot(3b)^2}{2\,!\cdot(2a)^6}+\frac{4\cdot 5\cdot 6\cdot(3b)^3}{3\,!\cdot(2a)^7}$$
$$+\cdots+\frac{4\cdot 5\cdot 6\cdot\cdots(r+3)(3b)^r}{r\,!\cdot(2a)^{r+4}}+\cdots,$$

So, $$(x+y)^{\frac{1}{2}}=x^{\frac{1}{2}}+\frac{1}{2}\cdot x^{-\frac{1}{2}}y+\frac{1}{2}\cdot\frac{{}^-1}{2}\cdot\frac{1}{2\,!}x^{-\frac{3}{2}}y^2$$
$$+\frac{1}{2}\cdot\frac{{}^-1}{2}\cdot\frac{{}^-3}{2}\cdot\frac{1}{3\,!}x^{-\frac{5}{2}}y^3+\cdots$$
$$+\frac{1}{2}\cdot\frac{{}^-1}{2}\cdot\frac{{}^-3}{2}\cdot\frac{{}^-5}{2}\cdots\frac{3-2r}{2}\cdot\frac{1}{r\,!}x^{-\frac{2r-1}{2}}y^r+\cdots$$
$$=x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}y-\frac{1}{8}x^{-\frac{3}{2}}y^2+\frac{1}{16}x^{-\frac{5}{2}}y^3-\frac{5}{128}x^{-\frac{7}{2}}y^4$$
$$+\cdots\pm\frac{1\cdot 3\cdot 5\cdot 7\cdots(2r-3)}{2^r\cdot r\,!}x^{-\frac{2r-1}{2}}y^r\mp\cdots.$$

NOTE. If n be a positive integer, the series ends with the $(n+1)$th term, since the coefficients of the following terms become 0; but if n be a negative integer, or a fraction, positive or negative, the series does not end, and is infinite.

§ 10. FINITE DIFFERENCES.

If there be any series of numbers, and if a second series be formed by subtracting each term of the first series from that which follows it, in order; a third series, by subtracting each term of the second series from that which follows it, and so on; then the terms of the second series are called the *differences of the first order*, or *first differences;* the terms of the third series are *the differences of the second order*, or *second differences;* and so on.

E.g., if	1,	4,	9,	16,	25,	36,	··· be a series,
then	3,	5,	7,	9,	11,	13,	··· are the first differences,
	2,	2,	2,	2,	2,	2,	··· are the second differences,
	0,	0,	0,	0,	0,	0,	··· are the third differences.
So, if	a,	b,	c,	d,	e,		··· be any series,
then	a_1,	b_1,	c_1,	d_1,	e_1,		··· are the first differences,
	a_2,	b_2,	c_2,	d_2,	e_2,		··· are the second differences,
	a_3,	b_3,	c_3,	d_3,	e_3,		··· are the third differences,

and so on:

wherein $a_1 = b - a, \quad b_1 = c - b, \quad c_1 = d - c, \quad \cdots,$

$a_2 = b_1 - a_1, \quad b_2 = c_1 - b_1, \quad c_2 = d_1 - c_1, \quad \cdots,$

and so on.

The series $a, a_1, a_2, a_3, \cdots$ is the *auxiliary* series; and the object of the theorems that follow is to show how to find one term and the sum of any number of terms of the principal series. by aid of the auxiliary series.

Theor. 22. *If* $\mathrm{a, b, c, d, e}, \cdots$ *be any series, and* $\mathrm{a, a_1, a_2, a_3}, \cdots$ *its auxiliary series, and if* $\mathrm{t_n}$ *be the* $(\mathrm{n}+1)$*th term of the principal series; then*

$$\mathrm{t_n = a + na_1 + \frac{n(n-1)}{2!} a_2 + \cdots + na_{n-1} + a_n}.$$

1. *The law is true when* $\mathrm{n} = 1$, 2, *and* 3.

For $\because a_1 = b - a,$ [df.

$\therefore b = a + a_1.$ Q.E.D. [$n = 1$

So, $b_1 = a_1 + a_2,\quad c = b + b_1,\quad d = c + c_1,$

$\therefore\ c, = b + b_1,$

$= a + 2a_1 + a_2.$ Q.E.D. [$n = 2$

So, $c_1 = a_1 + 2a_2 + a_3,$

$\therefore\ d, = c + c_1,$

$= a + 3a_1 + 3a_2 + a_3.$ Q.E.D. [$n = 3$

2. *If the law be true when* $\mathrm{n} = \mathrm{k}$, *it is true when* $\mathrm{n} = \mathrm{k} + 1$.

For, let q, r be the $(k+1)$th and $(k+2)$th terms of the principal series,

then $\because\ q = a + \mathrm{c}_1k \cdot a_1 + \mathrm{c}_2k \cdot a_2 + \cdots + \mathrm{c}_rk \cdot a_r + \cdots + a_k,$ [hyp.

and $q_1 = a_1 + \mathrm{c}_1k \cdot a_2 + \cdots + \mathrm{c}_{r-1}k \cdot a_r + \cdots + \mathrm{c}_1k \cdot a_k + a_{k+1},$

$\therefore\ r, = q + q_1,$

$= a + \mathrm{c}_1(k+1) \cdot a_1 + \mathrm{c}_2(k+1) \cdot a_2 + \cdots + \mathrm{c}_r(k+1) \cdot a_r$

$+ \cdots + \mathrm{c}_1(k+1) \cdot a_k + a_{k+1}.$ Q.E.D. [IV. th. 3, cr. 2

3. *The law is true universally.*

For $\because$ it is true when $n = 3$, [1

$\therefore$ it is true when $n = 3 + 1 = 4$, [2

$\therefore$ it is true when $n = 4 + 1 = 5$,

$\therefore$ it is true when $n = 6, 7, 8, \cdots$. Q.E.D.

NOTE 1. The reader may compare this proof with the third proof of the binomial theorem. [V. th. 1, nt. 2

NOTE 2. This theorem is of special value when the auxiliary series is short, ending in zeros.

E.g., of the series 1, 8, 27, 64, 125, ⋯, the auxiliary series is 1, 7, 12, 6, 0, 0, ⋯;

and $$\mathrm{T}_{10} = 1 + 9 \cdot 7 + \frac{9 \cdot 8}{2} \cdot 12 + \frac{9 \cdot 8 \cdot 7}{6} \cdot 6 = 1000.$$

So, of the series 7, 16, 27, 40, 55, ⋯, the auxiliary series is 7, 9, 2, 0, 0, ⋯;

and $$\mathrm{T}_{10} = 7 + 9 \cdot 9 + \frac{9 \cdot 8}{2} \cdot 2 = 160,$$

$$\mathrm{T}_n = 7 + (n-1)\,9 + \tfrac{1}{2}(n-1)(n-2)\,2 = n(n+6).$$

THEOR. 23. *If the terms of a series be like entire functions of their number in the series, the auxiliary series ends with that term whose number is one greater than the degree of the function.*

Let the general term of the series be

$$\text{T}_n, = \text{A} + \text{B}n + \text{C}n^2 + \cdots + \text{K}n^m: \qquad [m \text{ a pos. integer}$$

then the general term T_n' of the series of first differences $a_1, b_1, c_1, \cdots,$ is

$$\begin{aligned}\text{T}_{n+1} - \text{T}_n, &= \text{B} + \text{C}(\overline{n+1}^2 - n^2) + \cdots + \text{K}(\overline{n+1}^m - n^m)\\ &= \text{B} + \text{C}(2n+1) + \cdots + \text{K}(m \cdot n^{m-1} + \cdots),\end{aligned}$$

which contains no higher power of n than n^{m-1}.

So, the general term T_n'' of the series of second differences $a_2, b_2, c_2, \cdots,$ is $\text{T}_{n+1}' - \text{T}_n'$, and contains no higher power of n than n^{m-2}; $\cdots$.

So, the general term of the series $a_{m-1}, b_{m-1}, c_{m-1}, \cdots,$ contains only the first power of m.

So, the general term of the series $a_m, b_m, c_m, \cdots,$ is free from m, *i.e.*, is constant,

and all the subsequent series, $a_{m+1}, b_{m+1}, \cdots, \cdots,$ consist of zeros. Q.E.D.

THEOR. 24. *If the terms of a series be like entire functions of their number in the series, the form of these functions is identical with that found by aid of the auxiliary series.*

For $\because$ $\text{A} + \text{B}n + \text{C}n^2 + \cdots + \text{K}n^m$

$$= a + (n-1)a_1 + \frac{(n-1)(n-2)}{2!}a_2 + \cdots$$

$$+ \frac{(n-1)(n-2)\cdots(n-m)}{m!}a_m$$

for all integral values of n, [ths. 22, 23

$\therefore$ these two functions, each an entire function of n of the mth degree, are equal for more than m values of the variable n,

$\therefore$ they are identical. Q.E.D. [XI. th. 4, cr. 3

THEOR. 25. *If* a, b, c ⋯ l *be* n *terms of any series, and* a, a_1, a_2, a_3 ⋯ *its auxiliary series, then*

$$a+b+c+\cdots+l=na+\frac{n(n-1)}{2!}a_1+\frac{n(n-1)(n-2)}{3!}a_2+\cdots.$$

For, from the given series form a new series,

$$0,\ a,\ a+b,\ a+b+c,\ \cdots,\ a+b+c+\cdots+l,$$

wherein the $(n+1)$th term is the sum of the first n terms of the given series;

then ∵ the first differences of the new series are the terms of the given series,

the second differences of the new series are the first differences of the given series;

and so on;

∴ the auxiliary series of the new series is $0, a, a_1, a_2, a_3, \cdots$,

and its $(n+1)$th term is

$$0+na+\frac{n(n-1)}{2!}a_1+\frac{n(n-1)(n-2)}{3!}a_2+\cdots.$$

Q.E.D. [th. 22

§ 11. INTERPOLATION.

IF for a series of values of a variable (the arguments) there be a corresponding series of values of some function of that variable, the insertion of intermediate values of the function corresponding to intermediate values of the variable is *interpolation.*

PROB. 10. TO INTERPOLATE VALUES BETWEEN THE TERMS OF A GIVEN SERIES.

(*a*) *The form of the function known: Apply the law of formation, as shown by the form of the function of* n.

E.g., of the series 1, 4, 9, 16, ⋯, the $(2\frac{1}{2})$th term is $(2\frac{1}{2})^2, = 6\frac{1}{4}$.

So, of the series 1, 4, 7, 10, 13, ⋯, the $(3\frac{1}{3})$th term is $7+\frac{1}{3}\cdot 3, = 8$.

So, of the series 1, 4, 16, 64, ⋯, the $(3\frac{1}{2})$th term is $1\cdot 4^{\frac{5}{2}}, = 32$.

(*b*) *The arguments equidifferent, and the form of the function unknown:*

From the given series form the auxiliary series, and find the n*th term of the given series by aid of the formula of theor.* 22.

Assume the law of formation to be that shown in the form of the n*th term, and get intermediate terms by the application of this law, as in case* (*a*).

E.g., of the series 1, 3, 6, 10, 15, 21, ⋯, the auxiliary series is 1, 2, 1, 0;

the nth term is

$$1 + 2(n-1) + \tfrac{1}{2}(n-1)(n-2), = \tfrac{1}{2}n(n+1);$$

and the $(2\frac{1}{2})$th term is $\frac{1}{2}\cdot 2\frac{1}{2}\cdot 3\frac{1}{2}, = 4\frac{3}{8}$.

So, of the series 1, 1.414, 1.732, 2, 2.236, 2.450, the values of the square roots of 1, 2, 3, 4, 5, 6, correct to three decimal places, the auxiliary series is

1, .414, −.096, .046, −.028, .02, ⋯;

and the approximate value of $\sqrt{2\frac{1}{2}}$ is

$$1 + \tfrac{3}{2}.414 - \tfrac{3}{8}.096 - \tfrac{1}{16}.046 - \tfrac{3}{128}.028 = 1.581.$$

Note 1. This rule assumes that the law of formation of the series is that found by aid of the auxiliary series and the formula of theor. 22. The right to make this assumption appears as follows: if the auxiliary series terminates, the formula gives a law of formation by which the integral terms may be found, *viz.*, that the function be a rational integral function of the argument; and, since the function so found is a continuous function, by its aid intermediate terms may be got.

Whether the original series was got by this law does not appear; but as this is the simplest law made known by the data, and as this law does give the integral terms, it is assumed as the law of formation of intermediate terms.

If the given series consist of two terms a, b, then the auxiliary series is a, a_1, and the formula of interpolation for T_{n+1} is $a + na_1$, the ordinary formula of proportional parts in common use with arithmetical tables.

E.g., $\log 500 = 2.6990$, $\log 501 = 2.6998$,

and $\log 500.6 = 2.6990 + .6 \times .0008 = 2.6995$.

If the given series consist of three terms, a, b, c, then the auxiliary series is a, a_1, a_2, and the formula of interpolation for T_{n+1} is

$$a + na_1 + \frac{n(n-1)}{2} a_2.$$

E.g., $\log 150 = 2.1761$, $\log 160 = 2.2041$, $\log 170 = 2.2304$; then the auxiliary series is 2.1761, .0280, $-.0017$,

and $$\log 163 = 2.1761 + 1.3 \times .0280 - \frac{1.3 \times .3}{2} \times .0017,$$
$$= 2.2122.$$

So, if the given series have but four terms, five terms, and so on.

If the given series be infinite, the formula of interpolation is also infinite, and it is available when convergent, *i.e.*, when no term of the auxiliary series $a, a_1, a_2, \cdots$ exceeds a given finite limit; for since the series of coefficients $1 + n + \frac{n(n-1)}{2} + \cdots$ is convergent when n is positive [th. 21], so is the formula of interpolation $a + na_1 + \frac{1}{2}n(n-1)a_2 + \cdots$ convergent. [th. 8, cr.

When available, this formula is better adapted to computation than is the more general formula of case (c).

(c) *The arguments not equidifferent:*

Let $\text{x}_1, \text{x}_2, \text{x}_3, \cdots \text{x}_{m+1}$ *be any arguments not equidistant, and* $\text{y}_1, \text{y}_2, \text{y}_3, \cdots \text{y}_m$ *the corresponding values of the function, to interpolate a value of the function* y, *corresponding to a given argument* x*; compute* y *by the formula*

$$\begin{aligned} y = {} & \frac{(x-x_2)(x-x_3)\cdots(x-x_{m+1})}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_{m+1})} y_1 \\ & + \frac{(x-x_1)(x-x_3)\cdots(x-x_{m+1})}{(x_2-x_1)(x_2-x_3)\cdots(x_2-x_{m+1})} y_2 \\ & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + \frac{(x-x_1)(x-x_2)\cdots(x-x_m)}{(x_{m+1}-x_1)(x_{m+1}-x_2)\cdots(x_{m+1}-x_m)} y_{m+1}. \end{aligned}$$

For, assume y to be an entire function of x of the mth degree, and write

$$y = \text{A} + \text{B}x + \text{C}x^2 + \cdots + \text{M}x^m,$$

then is this function identical with that written above.

For $\because$ they are equal when $x = x_1$, when $x = x_2$, when $x = x_{m+1}$;

i.e., for $m+1$ values of x,

$\therefore$ they are identical. [XI. th. 4, cr.

E.g., if $x_1, x_2, x_3 = 150, 160, 180$,

and $y_1, y_2, y_3 = 2.1761,\ 2.2041,\ 2.2553$,

to interpolate a value of y corresponding to $x = 163$:

then $$y = \frac{3 \cdot {}^-17}{{}^-10 \cdot {}^-30} 2.1761 + \frac{13 \cdot {}^-17}{10 \cdot {}^-20} 2.2041 + \frac{13 \cdot 3}{30 \cdot 20} 2.2553$$

$$= 2.2122.$$

NOTE 2. When $x_1, x_2, \cdots$, have a constant difference 1, the formula of case (c) is equivalent to that of case (b).

For $\because$ each of these formulæ makes y an entire function of x of the mth degree,

and $\because$ both formulæ give the same value to y for more than m values of x,

$\therefore$ the two functions are identical. [XI. th. 4, cr.

NOTE 3. The principle of interpolation is illustrated graphically in the platting of curves by means of points. The abscissas of the points represent arguments; the corresponding ordinates represent the known values of the function; and any intermediate ordinate represents an intermediate value of the function. Graphically the interpolation is effected by joining the given points by the simplest smooth curve that can be drawn through them, and measuring the ordinate that corresponds to any given argument. The most reliable part of this curve is commonly that which is not too near either end.

§ 12. TAYLOR'S THEOREM.

Lemma. *If* $f(x+y)$ *be any finite continuous function of the sum* $x+y$ *for all values of that sum between* a *and* b, *then for all such values* $D_x f(x+y) = D_y f(x+y)$.

For if x be increased by h while y stands fast,

then $$D_x f(x+y) \equiv \lim \frac{f(x+y+h)-f(x+y)}{h}, \qquad \text{[df. deriv.}$$

and if y be increased by h while x stands fast,

then $$D_y f(x+y) \equiv \lim \frac{f(x+y+h)-f(x+y)}{h}.$$

$$\therefore\ D_x f(x+y) = D_y f(x+y). \qquad \text{[II. ax. 1}$$

Theor. 26. *If* $f(x+y)$ *be a continuous function of the sum* $(x+y)$ *that does not become infinite when* $y=0$, *its expansion in powers of* y *can contain no negative powers of* y.

For if possible let the expansion contain a term cy^{-m},

wherein c is independent of y;

then $\because\ cy^{-m} \doteq \infty$ when $y \doteq 0$,

$\therefore\ f(x+y) \doteq \infty$,

which is contrary to the hypothesis,

$\therefore$ this expansion can contain no negative powers of y.

Theor. 27. *If* $f(x+y)$ *and its successive derivatives be finite and continuous functions of the sum* $(x+y)$, *the expansion of* $f(x+y)$ *can contain no fractional power of* y.

For if possible let the expansion contain a term $cy^{n+\frac{p}{q}}$,

wherein c is free from y, n is a positive integer, and $\frac{p}{q}$ is a proper fraction;

then the $(n+1)$th derivative of this term as to y is $c'y^{\frac{p}{q}-1}$,

wherein c' is free from y, and $\frac{p}{q}-1$ is negative,

and $\because\ c'y^{\frac{p}{q}-1} \doteq \infty$ when $y \doteq 0$,

$\therefore\ f^{(n+1)}(x+y) \doteq \infty$,

which is contrary to the hypothesis;

$\therefore$ this expansion can contain no fractional powers of y.

NOTE. It is shown in the *theory of functions* that if a function of y and its y-derivatives be finite, continuous, and one-valued for all values of y smaller than a constant r, the function may be expanded to a series of rising integral powers of y that is convergent when y is smaller than r. This is equivalent to saying that $f(x+y)$ may be expanded to rising integral powers of y when $f(x+y)$, $f'(x+y)$, $\cdots$ are finite, continuous, and one-valued functions of $x+y$ from $y=0$ to $y=r$.

THEOR. 28. (TAYLOR'S THEOREM.) *If* $f(x+y)$ *be continuous, and if it be possible to expand this function in a series to positive integral powers of* y, *then*

$$f(x+y)=fx+\frac{y}{1}f'x+\frac{y^2}{2!}f''x+\frac{y^3}{3!}f'''x+\cdots+\frac{y^n}{n!}f^{(n)}x+\cdots,$$

wherein fx, $f'x$, $f''x$, $f'''x$ ... $f^{(n)}x$ ... *are what* $f(x+y)$ *and its successive derivatives become when* $y=0$.

For, put $f(x+y)\equiv A+By+Cy^2+Dy^3+\cdots+Ky^n+\cdots$,

wherein A, B, C, D, ... K, ... are finite and continuous functions of x, but free from y, and whose first derivatives as to x are all finite; [hyp.

then $A=fx$;

and $\because$ $D_x f(x+y)=D_x A+D_x B\cdot y+D_x C\cdot y^2+\cdots+D_x K y^n+\cdots$,

and $D_y f(x+y)=0+B+2Cy+3Dy^2+\cdots+nKy^{n-1}+\cdots$,

and $\because$ $D_x f(x+y)=D_y f(x+y)$, [lem.

$\therefore$ $D_x A+D_x B\cdot y+D_x C\cdot y^2+\cdots=B+2C\cdot y+3Dy^2+\cdots$ for all finite values of y. [th.18, cr.1

$\therefore$ $B=D_x A=f'x$,

$2C=D_x B=f''x$, $\therefore$ $C=\frac{1}{2}f''x$;

$3D=D_x C=\frac{1}{2}f'''x$, $\therefore$ $D=\frac{1}{3!}f'''x$ ...; and so on.

$$\therefore f(x+y)=fx+\frac{y}{1}f'x+\frac{y^2}{2!}f''x+\frac{y^3}{3!}f'''x+\cdots. \quad \text{Q.E.D.}$$

APPLICATIONS OF TAYLOR'S THEOREM.

By the use of Taylor's formula new methods, often simple, are found for the expansion of many expressions in series.

1. *The binomial formula.*

Let $f(x+y) \equiv (x+y)^n$;

then $\because\ fx = x^n,\ f'x = nx^{n-1},\ f''x = n(n-1)x^{n-2}, \ldots,$ [VIII. ths. 16, 14

$$\therefore\ (x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2\,!}x^{n-2}y^2 + \cdots,$$

a convergent series when $x > y$. [th. 10, nt. 2

2. *The logarithmic series.*

Let $f(x+y) \equiv \log_A(x+y)$;

then $\because\ fx = \log_A x,\ f'x = M_A x^{-1},\ f''x = -M_A x^{-2},\ f'''x = 2M_A x^{-3},$

$$\therefore\ \log_A(x+y) = \log_A x + M_A\left(\frac{y}{x} - \frac{y^2}{2x^2} + \frac{y^3}{3x^3} - \frac{y^4}{4x^4} + \cdots\right).$$

COR. *If* $x = 1$, *then* $\log_A(1+y) = M_A\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots\right)$;

a convergent series when $y < 1$;

So, when $y < 1$, $\log_A(1-y) = M_A\left(-y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \cdots\right)$.

3. *Maclaurin's formula.*

Let $fy,\ f0,\ f'0,\ f''0,\ \ldots$ be what $f(x+y),\ fx,\ f'x,\ f''x, \cdots$ become when x is 0;

then $$fy = f0 + f'0\cdot y + \frac{f''0}{2\,!}\cdot y^2 + \frac{f'''0}{3\,!}\cdot y^3 + \cdots.$$

4. *The exponential series.*

Let $fy \equiv A^y$;

then $\therefore\ f0 = A^0 = 1,\quad f'0 = A^0 : M_A = M_A^{-1},\quad f''0 = A^0 : M_A^{\ 2} = M_A^{-2},$

$$\therefore\ A^y = 1 + M_A^{-1}\cdot y + \frac{M_A^{-2}y^2}{2\,!} + \frac{M_A^{-3}y^3}{3\,!} + \cdots,$$

wherein $M_A^{-1},\ M_A^{-2},\ \cdots,\ = \log_e A,\ (\log_e A)^2,\ \cdots.$

COR. 1. $e^y = 1 + y + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \cdots$;

a convergent series for all finite values of y, [th. 11, nt. 2

and $$e = e^1 = 1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \cdots = 2.7182818\cdots.$$

So, $A^x = e^{x\log_e A} = 1 + x\log_e A + \frac{(x\log_e A)^2}{2\,!} + \frac{(x\log_e A)^3}{3\,!} + \cdots,$ [IX. th.

§ 13. COMPUTATION OF LOGARITHMS.

THEOR. 29. *If* N *be any number greater than* 1, A *any positive base, and* M_A *the modulus of the system; then*

$$\log_A N = \log_A(N-1)$$
$$+ 2M_A\left(\frac{1}{2N-1}+\frac{1}{3(2N-1)^3}+\frac{1}{5(2N-1)^5}+\cdots\right).$$

For take $y=\frac{1}{2N-1}$, whence $\frac{1+y}{1-y}=\frac{N}{N-1}$;

then $\because \log_A(1+y)=M_A\,(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\cdots)$.

and $\log_A(1-y)=M_A\,(-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}\cdots)$. [th. 28 ap. 2

$\therefore \log_A\frac{1+y}{1-y}=2M_A(y+\frac{y^3}{3}+\frac{y^5}{5}+\frac{y^7}{7}+\cdots)$. [IX. th. 6

and $\because \log_A N-\log_A(N-1)=\log_A\frac{N}{N-1}=\log_A\frac{1+y}{1-y}$;

$\therefore \log_A N=\log_A(N-1)+2M_A(y+\frac{y^3}{3}+\frac{y^5}{5}+\frac{y^7}{7}+\cdots)$

$$=\log_A(N-1)+2M_A\left(\frac{1}{2N-1}+\frac{1}{3(2N-1)^3}+\cdots\right).$$

This series is convergent if $N>1$. [th. 11 nt. 4

COR. 1. *If* $N-1$ *be any positive fraction, however small,*

then $\log_A(N-1)=\log_A N-2M_A\left(\frac{1}{2N-1}+\frac{1}{3(2N-1)^3}+\cdots\right)$.

COR. 2. *If* $A=e$, *the Napierian base, then* $M_A=1$,

and $\log_e N=\log_e(N-1)+2\left(\frac{1}{2N-1}+\frac{1}{3(2N-1)^3}+\cdots\right)$.

PROB. 11. TO COMPUTE A TABLE OF NAPIERIAN LOGARITHMS:

Beginning with 2, compute the logarithm of every prime number in order. [th. 29 cr. 2

For the logarithms of composite numbers, add together the logarithms of their factors. [IX. th. 6

E.g., $\log_e 2 = \log_e 1 + 2\left(\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \cdots\right).$

$$\begin{array}{r|rcr}
 & \log 1 & = & 0 \\
9 & .66666667 : 1 & = & .66666667 \\
9 & 7407407 : 3 & = & 2469136 \\
9 & 823045 : 5 & = & 164609 \\
9 & 91449 : 7 & = & 13064 \\
9 & 10161 : 9 & = & 1129 \\
9 & 1129 : 11 & = & 103 \\
9 & 125 : 13 & = & 10 \\
 & 14 : 15 & = & \underline{1} \\
= & & & .693147
\end{array}$$

So, $\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \cdots\right) = 1.098612.$

So, $\log_e 4 = 2 \cdot \log_e 2 \qquad = 1.386294.$

So, $\log_e 5 = \log_e 4 + 2\left(\frac{1}{9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \cdots\right) = 1.609438.$

So, $\log_e 10 = \log_e 2 + \log_e 5 = 2.302585.$

PROB. 12. TO COMPUTE A TABLE OF COMMON LOGARITHMS:

For prime numbers, multiply the Napierian logarithms, found as above, by .43429448.

For composite numbers, add the logarithms of their factors.

For, $\log_{10} N = \log_e N : \log_e 10,$ [IX. th. 8

$= \log_e N : 2.302585$

$= \log_e N \times .43429448,$

wherein $M_{10}, = .43429448$, is the reciprocal of 2.302585.

E.g., $\log_{10} 2 = .693147 \times .43429448 = .301030.$ [pr. 9

So, $\log_{10} 3 = 1.098612 \times .43429448 = .477121.$

NOTE. The work is further shortened by interpolation.

§ 14. EXAMPLES.

§ 1.

… 4. Find the last term and the sum of 5 terms, 20 terms, 35 terms, 50 terms, $2n$ terms, $2n+1$ terms, of the series:

1. The natural numbers; the odd numbers; the even numbers.
2. The numbers of the form $r+kx$ wherein r, k are constant integers and x a variable integer.
3. The distances passed over in successive seconds by a falling body, starting from rest (16.1, 48.3, 80.5, … feet, or 4.9, 14.7, 24.5, … meters).
4. $1, {}^-2, {}^+3, {}^-4, \cdots$; $1, {}^-3, {}^+5, {}^-7, {}^+9, \cdots$; $3, 2\frac{2}{3}, 2\frac{1}{3}, \cdots$.
5. One hundred stones are placed in a line on the ground a meter apart, and a basket is placed a meter from the first stone; how many kilometers must a man run, who, starting from the basket, picks up all the stones, one by one, and returns to the basket each time he picks up a stone?

… 8. Find the five elements of the arithmetic progressions:

6. $1, 3, 5, \cdots 99$; $1, 3, 5, \cdots 2k-1$; $4+5+6+\cdots=5350$.
7. $5\cdots 7$ means $\cdots 75$; $3\cdots 11$ means $\cdots - 11$; $2\frac{1}{2}\cdots 3$ means $\cdots 20$.
8. $\cdots 5$ terms $\cdots 19 \cdots 7$ means $\cdots 67$; $1, x, \cdots 4x, 19$; $1+\cdots+50=204$;
9. Fill out the arithmetic progressions:
   $0+\cdots+3\cdots+4=10, =18, =2(4k+1)$. [$k$ any integer
10. Find the distances passed over by a body falling from rest in successive quarter seconds; and in successive periods of 5 seconds. [ex. 3
11. A stone thrown into the air took 5 seconds to rise and fall to the same level; how high was it thrown? [ex. 3
12. Find the condition that a, b, c may be the pth, qth, rth terms of an arithmetic progression; if this condition be satisfied, and if a, b, c be positive integers, show that p, q, r may be the ath, bth, cth terms of an arithmetic progression, and that the product of the common differences of the two progressions is unity.

13. Divide unity into 4 parts in arithmetic progression, such that the sum of their cubes shall be $\frac{1}{10}$.

14. The interior angles of a rectilinear figure are in arithmetic progression; the least angle is 120° and the common difference 5°; find the number of sides.

15. A three-digit number is 26 times the sum of its digits; the digits are in arithmetic progression; if 396 be added to the number, the digits are reversed: find the number.

16. At 4 P.M., A, riding 4 miles an hour, is 11 miles ahead of B; B increases his speed regularly $\frac{1}{4}$ of a mile every hour, and has ridden since starting at 11 P.M. the day before, $72\frac{1}{4}$ miles; when did A pass B, and when will B pass A?

§ 2.

... 19. Find the last term, and the sum, of 10 terms, n terms, ∞ terms, of the series:

17. The integral powers of ± 2; ± 3; $\pm k$; $\pm\frac{1}{2}$; $\pm\frac{1}{3}$; $\pm\frac{1}{k}$. [$k>1$

18. $5-\frac{1}{2}+\frac{1}{20}\cdots$; $\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+\cdots$; $1+(\ )+\frac{1}{16}+\cdots$.

19. $1+(\ )+(\ )-\frac{3}{4}+\cdots$; $.672672\cdots$; $\frac{2}{5}+\frac{3}{5^2}+\frac{2}{5^3}+\frac{3}{5^4}+\cdots$.

20. A man invests \$100 half-yearly in stocks that pay 3 per cent half-yearly dividends, and invests the dividends as they are received; how much will he have invested at the end of 10, 20, 30 years?

21. A man at 20 insures his life for \$2000, paying therefor a premium of \$20 half-yearly; what is the gain or loss to the insurance company if he die at 30, 40, 50, 60, 70, estimating that it costs the company 10 per cent of its premiums to collect and care for them, and that money is worth 5 per cent per annum?

22. Show that $\sqrt{.444\cdots}=.666\cdots$; $\sqrt[3]{2.370370\cdots}=1.333\cdots$.

23. Find four geometric means between 1 and 32; two between .1 and 100; three between $\frac{1}{9}$ and 9; three between 2 and $\frac{1}{8}$.

24. The sum of three numbers in geometric progression is 13, and the product of the mean and the sum of the extremes is 30; what are the numbers?

25. Show that, if n geometric means lie between a and c, their product is $(ac)^{\frac{n}{2}}$.

26. If the common ratio of a geometric progression be less than $\frac{1}{2}$, prove that every term is greater than the sum of all the terms that follow it.

27. What is the condition that a, b, c may be the pth, qth, rth terms of a geometric progression? If this condition be satisfied, and $\log_A a$, $\log_A b$, $\log_A c$ be positive whole numbers a', b', c', show that A^p, A^q, A^r are the a'th, b'th, c'th terms of a geometric progression.

28. If there be an infinite number of infinite decreasing geometric progressions, wherein the ratio is common, and the first term of each is the nth term of that just before it, show that their sum is $a:(1-r)(1-r^{n-1})$.

29. There are two infinite decreasing geometric progressions, each beginning with 1, whose sums are s, s': prove that the sum of the series formed by multiplying their corresponding terms is $ss':(s+s'-1)$.

§ 3.

30. Continue in both directions the harmonic progressions: $2, 3, 6$; $3, 4, 6$; $1, 1\frac{1}{6}, 1\frac{2}{5}$; to five terms, to n terms.

31. The difference of two numbers is 8 and their harmonic mean is $1\frac{4}{5}$; what are the numbers?

32. What is the condition that a, b, c be the pth, qth, rth terms of a harmonic progression?

33. If a, b, c, ... be in geometric progression, and $a^p = b^q = c^r = \ldots$, then p, q, r, ... are in harmonic progression.

34. Prove that the arithmetic, geometric, and harmonic means of two numbers greater than unity are in descending order of magnitude.

§ 4.

... 53. Determine which of the series are convergent:

35. $\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\cdots;\quad 1+\frac{3}{2}+\frac{3^2}{2^2}+\cdots;\quad \frac{1}{100}+\frac{2}{100}+\frac{3}{100}+\cdots.$

36. $1+\frac{m+p}{1}\cdot\frac{3}{5}+\frac{m+2p}{2}\cdot\frac{3^2}{5^2}+\frac{m+3p}{3}\cdot\frac{3^3}{5^3}+\cdots.$

37. $1+\frac{2}{1}\cdot\frac{1}{2}+\frac{3}{2}\cdot\frac{1}{2^2}+\frac{4}{3}\cdot\frac{1}{2^3}+\cdots;\quad \frac{a+h}{b+k}\cdot\frac{1}{r}+\frac{a+2h}{b+2k}\cdot\frac{1}{r^2}+\cdots.$

38. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots;\quad \frac{1}{\sqrt{(1\cdot 2)}}+\frac{1}{\sqrt{(2\cdot 3)}}+\frac{1}{\sqrt{(3\cdot 4)}}+\cdots.$

39. $\frac{1}{1\cdot 3}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\cdots;\quad 1+\frac{3}{2}+\frac{4\cdot 5}{2\cdot 3}+\frac{5\cdot 6\cdot 7}{2\cdot 3\cdot 4}+\cdots.$

40. $\frac{1}{a(a+b)}+\frac{1}{(a+b)(a+2b)}+\frac{1}{(a+2b)(a+3b)}+\cdots.$

41. $\frac{5}{1\cdot 2\cdot 3}+\frac{6}{2\cdot 3\cdot 4}+\frac{7}{3\cdot 4\cdot 5}+\cdots;\quad \frac{1}{2\cdot 3}+\frac{2}{3\cdot 4}+\frac{3}{4\cdot 5}+\cdots.$

42. $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots;\quad 1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots.$

43. $1+\frac{1}{2}+\frac{1\cdot 3}{2\cdot 4}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}+\cdots;\quad \frac{2^2}{4!}+\frac{2^2\cdot 4^2}{6!}+\frac{2^2\cdot 4^2\cdot 6^2}{8!}+\cdots.$

44. Find s_5, and its bounds of error, in each of the above series.

45. Write the above series to powers of x so that x^n shall have for coefficient the nth term of the series, and determine the radius of convergence in each case.

§ 5.

46. $1-2+3-4+\cdots;\quad 1-1+1-1+\cdots.$

47. $a-b+c+a+b-c-a+b+c+a-b+c+\cdots.$

48. $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\cdots;\quad 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots;$

49. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots;\quad \frac{2+1}{2}-\frac{2^2+1}{2^2}+\frac{2^3+1}{2^3}-\cdots.$

50. $\frac{a-1}{1^2}-\frac{2(2a-1)}{2^2}+\frac{3(3a-1)}{3^2}-\cdots.$

51. $\frac{1}{1\cdot 2}-\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}-\cdots;\quad \frac{1}{1\cdot 3}-\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}-\cdots.$

52. $\frac{2}{3\cdot5}-\frac{3}{5\cdot7}+\frac{4}{7\cdot9}-\frac{5}{9\cdot11}+\cdots$; $1-\frac{1}{2}+\frac{1\cdot3}{2\cdot4}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}+\cdots$.

53. $1-\frac{1}{2^2}-\frac{1^2\cdot3}{2^2\cdot4^2}-\frac{1^2\cdot3^2\cdot5}{2^2\cdot4^2\cdot6^2}+\cdots$.

§ 6.

54. Write the series in §§ 4, 5 to rising powers of $x, x-1, x+1, x-2, x+2, x-1+i, x-2+i$, so that x^n, $(x-1)^n$, $\cdots$, shall have for coefficient the nth term of the series, and construct the circles of convergence of the resulting series.

§ 7.

55. Determine in advance from the character of the functions in Exs. 56–8, 66, 67, 69–71, what will be the radius of convergence of their expansions to rising powers of x.

§ 8.

$\cdots$58. Expand into series to rising powers of x, the fractions:

56. $\frac{1}{3-2x}$; $\frac{3x-2}{(x-1)(x-2)(x-3)}$; $\frac{5-10x}{2-x-3x^2}$; $\frac{1}{1-x+x^2}$.

57. $\frac{x}{(1-x)(1-ax)}$; $\frac{1-x^3}{2-2x-x^2}$; $\frac{1}{(1+x)^2}$; $\frac{1}{(1+x)^n}$.

58. $\frac{1}{(1+x)^3}$; $\frac{1}{a^2+ax+x^2}$; $\frac{1}{a^3+a^2x+ax^2+x^3}$; $\frac{a-x}{a^n-x^n}$; $\frac{a^2-x^2}{a^{2n}-x^{2n}}$.

$\cdots$65. Find the scale and sum of the recurring series:

59. $4+9x+21x^2+51x^3+\cdots$; $1+3x+2x^2-x^3-\cdots$.

60. $1+3x+11x^2+43x^3+\cdots$; $1+2x+3x^2+4x^3+\cdots$.

61. $1+3x+6x^2+10x^3+15x^4+21x^5+\cdots$;
$1-x+x^4-x^5+x^8-x^9+\cdots$.

62. $1+3x+5x^2+7x^3+\cdots$; $1+x+2x^2+2x^3+3x^4+3x^5+\cdots$.

63. $1+1+\frac{3}{4}+\frac{1}{2}+\frac{5}{16}+\cdots$; $\frac{1}{6}+\frac{5}{36}x+\frac{19}{216}x^2+\frac{65}{1296}x^3+\cdots$.

64. $\frac{(a+1)-a}{a(a+1)}+\frac{(a+1)^2-a^2}{a^2(a+1)^2}x+\frac{(a+1)^3-a^3}{a^3(a+1)^3}x^2+\cdots$.

65. $1+2x+\frac{5}{2}x^2+\frac{5}{2}x^3+\frac{35}{16}x^4+\frac{7}{4}x^5+\cdots$; $4+3+\frac{7}{3}+\frac{17}{9}+\cdots$.

$\cdots$67. Expand into series to rising powers of x, the surds:

66. $\sqrt{(a-x)}$; $\sqrt{(a^3-x^3)}$; $\sqrt{(a^2+ax+x^2)}$; $\sqrt[3]{(a^3-x^3)}$.

67. $\dfrac{1}{\sqrt{(a-x)}}$; $\dfrac{1}{\sqrt{(a^3-x^3)}}$; $\dfrac{1}{\sqrt{(a^2+ax+x^2)}}$; $\dfrac{1}{\sqrt[3]{(a^3-x^3)}}$.

68. Find the values correct to four decimal places of:
$\sqrt{3}$; $\sqrt{5}$; $\sqrt[3]{9}$; $\sqrt[3]{31}$; $\sqrt[4]{17}$; $\sqrt[4]{80}$; $\sqrt[5]{33}$; $\sqrt[5]{240}$; $\sqrt[6]{720}$.

... 71. Resolve into sums of partial fractions, the fractions:

69. $\dfrac{3x-2}{(x-1)(x-2)(x-3)}$; $\dfrac{5+6x}{(1-3x)^2}$; $\dfrac{1+4x+x^2}{(1-x)^4}$.

70. $\dfrac{ax^2}{1-2x+x^2}$; $\dfrac{(a-b)x}{x^2-(a-b)x+ab}$; $\dfrac{x^2}{(x-a)(x-b)(x-c)}$.

71. $\dfrac{x^3}{x^3-4x^2+5x-2}$; $\dfrac{x^2-x-3}{x(x^2-4)}$; $\dfrac{2x^2-7x+1}{x^3+1}$.

... 75. Resolve each term into its partial fractions, and by aid of the series so formed find the sum of the series:

72. $\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\cdots\dfrac{1}{n(n+1)}$; $\dfrac{1}{1\cdot4}+\dfrac{1}{2\cdot5}+\cdots\dfrac{1}{n(n+3)}$;

73. $\dfrac{1}{1\cdot3}+\dfrac{1}{3\cdot5}+\dfrac{1}{5\cdot7}+\cdots$; $\dfrac{4}{1\cdot2\cdot3}+\dfrac{5}{2\cdot3\cdot4}+\dfrac{6}{3\cdot4\cdot5}+\cdots$.

74. $\dfrac{1}{1\cdot2\cdot3\cdot4}+\dfrac{1}{2\cdot3\cdot4\cdot5}+\cdots$, $\dfrac{1}{1\cdot3\cdot5\cdot7}+\dfrac{2}{3\cdot5\cdot7\cdot9}+\cdots$.

75. $$\frac{x}{(1+x)(1+ax)}+\frac{ax}{(1+ax)(1+a^2x)}+\frac{a^2x}{(1+a^2x)(1+a^3x)}+\cdots+\frac{a^{n-1}x}{(1+a^{n-1}x)(1+a^nx)}.$$

76. Resolve into partial fractions the fraction:
$\dfrac{x^p}{(x-a_1)(x-a_2)\cdots(x-a_n)}$, and show that when $p<n$,
$$\frac{a_1^{p-1}}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)}+\frac{a_2^{p-1}}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)}+\cdots+\frac{a_n^{p-1}}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}=0.$$

... 81. Revert the series:

77. $y=x^2+x$; $y=4x-x^2$; $y=13x-6x^2$.

78. $y=6x^3+x$; $y=3x-x^3$; $y=x^3-15x$.

79. $y=20x+10x^2-x^4$; $y=x+x^2+x^3+x^4+\cdots$.

80. $y=ax+bx^2$; $y=ax+bx^2+cx^3$; $y=ax^2+bx^4+cx^6+dx^8$.

81. $y=(a-x)^{-1}$; $y=x(a-x)^{-1}(b-x)^{-1}$; $y^2=a-x^2$.

82. Show that, if $y = ax^n + bx^{n+p} + cx^{n+2p} + \cdots$,
then $x = \text{A}y^{\text{N}} + \text{B}y^{\text{N+P}} + \text{C}y^{\text{N+2P}} + \cdots$,
wherein $\text{N} = 1 : n$, $\text{P} = \text{N}p$, $\text{A} = a^{-\text{N}}$, $\text{B} = -\text{NA}^{n+p+1}b$;
show that $n, p, a, \cdots$ are the same functions of $\text{N}, \text{P}, \text{A}, \cdots$ as $\text{N}, \text{P}, \text{A}, \cdots$ are of $n, p, a, \cdots$.

§ 9.

··· 87. Expand to six terms, and write the general term of:

83. $\sqrt{(1-x)}$; $\sqrt[3]{(1-x)}$; $\sqrt[3]{(1-x^3)}$; $\sqrt[4]{(1+x)}$; $\sqrt[r]{(1-px)}$.

84. $(1-x)^{-\frac{1}{2}}$; $(1-x)^{-\frac{1}{3}}$; $(1+x)^{-\frac{2}{3}}$; $(1-x)^{-2}$; $(1-x)^{-4}$.

85. $(1+x)^{-n}$; $(1-x^2)^{-\frac{2}{3}}$; $(a^2-x^2)^{-\frac{1}{2}}$; $(1+x+x^2+\cdots)^n$.

86. $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$; $\dfrac{(1-2x)^3}{(1-3x^2)^4}$; $\dfrac{x^n}{(1-x)^{2n}}$; $(1+x)^x$; $\dfrac{(1+x)^{\frac{1}{x}}}{(1+x^{-1})^x}$.

87. $(1+3x+5x^2+\cdots)^4$; $(1+2x+3x^2+4x^3+\cdots)^n$.

88. Find the radius of convergence of the series in Exs. 56–87.

89. By aid of the binomial theorem compute the values, correct to 5 decimal places, of the surds in Ex. 68.

§ 10.

··· 92. Find the sum of the first 5 terms, 20 terms, n terms of:

90. $1+2+3+\cdots$, $1^2+2^2+3^2+\cdots$, $1^3+2^3+3^3+\cdots$.

91. $1+3+5+\cdots$, $1^2+3^2+5^2+\cdots$, $1^3+3^3+5^3+\cdots$.

92. $a+\overline{a+d}+\overline{a+2d}+\cdots$, $a^2+\overline{a+d}^2+\overline{a+2d}^2+\cdots$,
$a^3+\overline{a+d}^3+\overline{a+2d}^3+\cdots$.

93. Find the series of values that $x^4-5x^3+4x^2-3x-8$ takes when $x=1, 2, 3, \cdots$, and plat the function.
Find the sum of 5 terms, 20 terms, n terms of this series.

94. Find an entire function of x that shall take the respective values 4, 6, 10, when $x = 1, 2, 3$.

95. Find the 5th term, 20th term, nth term of the series of figurate numbers: 1, 1, ···; 1, 2, 3, ···; 1, 3, 6, 10, ···; 1, 4, 10, 20, 35, ···; 1, 5, 15, 35, 70, 126, ···.

96. If shot be piled in a triangular pyramid, find how many shot there are in the 5th, 20th, nth courses, counting down from the top; and how many shot altogether in the 5 upper courses, 20 courses, n courses.

97. So, if piled in a square pyramid.